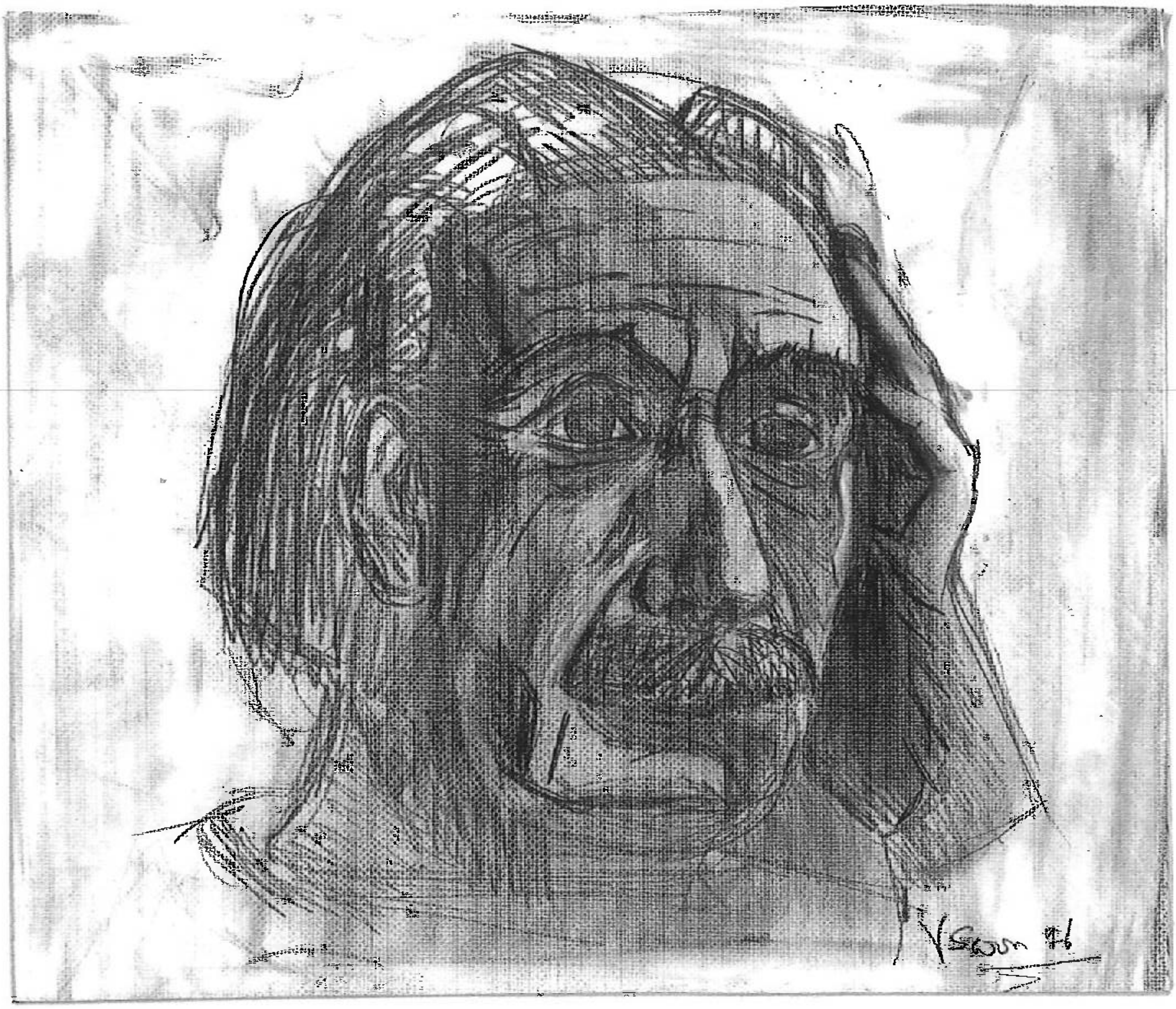


G.R. ①

Bianchi Identity:

Einstein Tensor

$\nabla_{[a} R_{bc]d}{}^{e} = 0$

ie $\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0$

contract c with e (symms.):

$\nabla_a R_{bd} - \nabla_b R_{ad} + \nabla_c R_{abd}{}^c = 0$

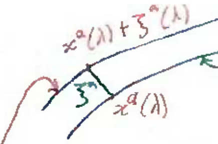
then hit with g^{bd} on left – can go straight through ∇_a as metric connection.

$\rightarrow \nabla_a R - \nabla_b R_a{}^b + \nabla_c R_{ab}{}^{bc} = 0$

$\Rightarrow \boxed{\left(R_a{}^b - \frac{1}{2}\delta_a{}^b R\right)_{;b} = 0}$ $-\nabla_c R_a{}^c$

ie $\boxed{G_a{}^b{}_{;b} = 0}$

Geodesic Deviation



Take two neighbouring geodesics: with connecting vector ξ^a

this has eq'n: $\frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\,(\text{at } x)\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$

this has eq'n: $(\ddot{x}^a + \ddot{\xi}^a) + \Gamma^a{}_{bc}\big|_{\text{at } x+\xi}(\dot{x}^b + \dot{\xi}^b)(\dot{x}^c + \dot{\xi}^c) = 0$

expand to $O(\xi)$:

$\ddot{x}^a + \ddot{\xi}^a + \left(\Gamma^a{}_{bc}\big|_{\text{at } x} + \xi^d \partial_d \Gamma^a{}_{bc}\big|_{\text{at } x}\right)(\dot{x}^b + \dot{\xi}^b)(\dot{x}^c + \dot{\xi}^c) = 0$

$\lambda = s$!

work in normal coordinates: $\Gamma^a{}_{bc} = 0$ so $\ddot{x}^a = 0$, get:

$\ddot{\xi}^a + \xi^d \Gamma^a{}_{bc,d}\dot{x}^b \dot{x}^c = 0$ but $\frac{D^2}{Ds^2}\xi^a = \frac{d^2\xi^a}{ds^2} + \Gamma^a{}_{bc,d}\xi^c \frac{dx^b}{ds}\frac{dx^d}{ds}$

and, in normal coords, $R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d}$ so:

$\boxed{\frac{D^2\xi^a}{Ds^2} + R^a{}_{bcd}\,\xi^b \frac{dx^c}{ds}\frac{dx^d}{ds} = 0}$ GEODESIC DEVIATION

$u^e \nabla_e (u^f \nabla_f \xi^a)$ where u^e is tang. vec to geodesic

Matter

Stress-energy tensor: $T^{ab} = \rho_0 u^a u^b$ (4-vel of dust cloud; ρ_0 in rest frame) for dust ie Pressure = 0

– flux of a-momentum through surface of constant b.

eg T^{00} is energy flux through const. time surface ie. the energy density (conservation of energy and momentum)

In SR, including pressure $T^{ab} = \rho u^a u^b + P(\eta^{ab} + u^a u^b)$ (in rest frame)

eq'n of motion for free fluid is $\boxed{\partial_a T^{ab} = 0}$ – gives Navier-St. eq'n and mass conservation for non-rel limit!!

In G.R. change to covariant derivative: $\nabla_a T^{ab} = 0$

Energy is defined to be: $E = p_a v^a$ for particle with mom p^a and observer vel v^a – only at a point! (ie intersection of geodesics) – p^a and v^a live in the tangent space at the geodesic's intersection!

for const vector field over M^4, v^a, $j_a = T_{ab} v^b$ is energy current measured by observer moving with 4-vel v^a.

use div. thm on $\partial^a j_a = 0$ to get:

$\int_V d^3x\, j_0\big|_{\text{const } t} =$ mass-energy in V at time t

$\int_S dS_a j^a =$ flux of energy out of volume...

Schwarzschild Solution

Sph. symm, stationary, static → invariant under $t \to -t$

∴ angular part is flat and $g_{rr} = g_{rr}(r)$ only $\longrightarrow -r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ | no explicit time dep: ∃ Killing vector $\left(\frac{\partial}{\partial t}\right)^a$ | so no $dt\,dx^a$ cross terms.

So metric is: $ds^2 = e^A c^2 dt^2 - e^B dr^2 - r^2 d\Omega^2$ and must get $A(r)$ and $B(r)$

from the vacuum field eq'ns: $\boxed{R_{ab} = 0}$

– gives: $\boxed{ds^2 = \left(1 - \frac{2\mu}{r}\right)c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$

The metric is consequence of only: sph. symm, static, vacuum field eq'ns

⟹ if surround empty sphere with shell of mass, no change! as with Newt.

Actually don't need the static condition (Birkhoff's theorem)

Physical Interpretation of Coords

Time

Angles (are as usual) – a rod of proper length l emitting light from its ends is seen to subtend the coord. angles (from centre)

Radial distance: angular diameter distance ie if know l, see ϕ, θ then r is the apparent distance.

Schwarschild Geodesics

Lagrangian, $L = \alpha c^2 \dot{t}^2 - \frac{\dot{r}^2}{\alpha} - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = \frac{1}{\alpha c^2}$ (time like) $= 0$ (null)

First integrals: $k = \alpha \dot{t}$, $h = r^2\dot{\phi}$ for $\theta = \frac{\pi}{2}$

then for timelike: $\left(\frac{h r'}{r^2}\right)^2 + \frac{h^2}{r^2} = c^2k^2 - c^2\alpha + \frac{2\mu h^2}{r\, r^2}$ (not for photons)

So: $u'^2 + u^2 = \frac{c^2(k-1)}{h^2} + \frac{2\mu u c^2}{h^2} + 2\mu u^3$ (missing for photons)

$\Rightarrow \boxed{u'' + u = \frac{\mu c^2}{h^2} + 3\mu u^2}$

Newton. this is the GR correction term

ORBITAL PRECESSION: solution to Newtonian eq'n is $u = \frac{\mu c^2}{h^2}(1 + e\cos\theta)$

use Newtonian solution to get inhomogeneous term ie $3\mu u^2$ then a second approx issues from const. cos θ term because P.I. is $\phi \sin\phi$! ie not periodic.

– get $u = \frac{\mu c^2}{h^2}(1 + e\cos\phi + \phi' \sin\phi)$ $\left(\phi' = \frac{3\mu^2 c^2 \phi}{h^2}\right)$

$\simeq \frac{\mu c^2}{h^2}\left(1 + e\cos\left[\phi\left(1 - \frac{3\mu^2 c^2}{h^2}\right)\right]\right)$

So: orbit precesses $\Delta\phi = \frac{6\pi\mu c^2}{h^2}$ per period.

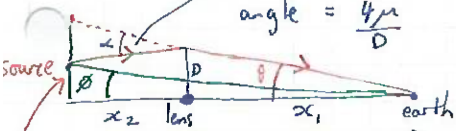
Comparing with Newtonian Circular Orbits: $u'' = 0$: Eliminate h, get $\left(\frac{\dot{\phi}}{u}\right)^2(u - 3\mu u^2) = \mu c^2 u^2$

$\Rightarrow r\dot{\phi}^2 = \frac{GM}{r(r - 3\mu)} \simeq \frac{GM}{r^2}$.

G.R. ②

Gravitational lensing

α is the bend angle $= \frac{4\mu}{D}$



x_2 lens x_1 ; x — angular diameter distances

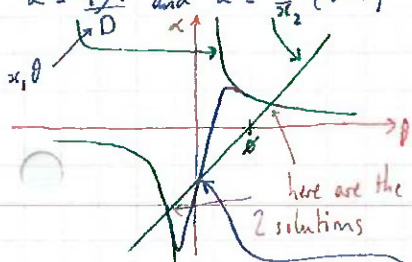
are lengths: $\theta x = \phi x + \alpha x_2$

$$\Rightarrow \theta^2 - \phi\theta - \frac{4\mu}{x}\left(\frac{x_2}{x_1}\right) = 0$$

→ 2 solutions, for point mass μ

can see better if plot $\alpha(\theta)$

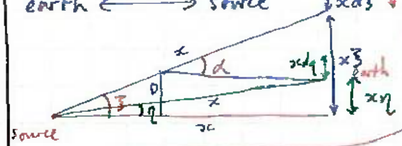
$\alpha = \frac{4\mu}{D}$ and $\alpha = \frac{x}{x_2}(\theta - \phi)$

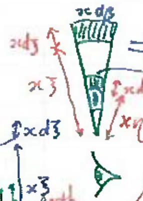


but ∃ a third one (for some geom) as for < D, $\mu \to 0$!! not pt mass so get blue line too

Light Amplification

Same diagram but earth ⟵⟶ source





$\Rightarrow \frac{d\alpha}{d\beta} = \frac{\eta}{\xi}$ transverse change in area (same no. of photons → things appear brighter)

Radial change: $\frac{x_1}{x_2}\frac{4\mu}{\xi} = x(\xi - \eta)$ for ξ, η ....

Now for $\xi + d\xi$ and $\eta + d\eta$, keep only $O(d\xi)$....

get $\frac{d\eta}{d\xi} = \frac{2\xi - \eta}{\xi}$ so $x d\xi \to x d\eta = x\left(\frac{2\xi - \eta}{\xi}\right) d\xi$

So total (transverse + radial) change in area is: $x^2 d\beta d\xi \to x^2 \frac{\eta}{\xi}\left(\frac{2\xi - \eta}{\xi}\right) d\beta d\xi$

Now say situation is prob symm coz angles are so small:

$\Rightarrow$ Amplification Factor $= \left|\frac{\theta}{\phi} \cdot \frac{\theta}{2\theta - \phi}\right|$

Have derived light amplification from area distortion - so also now know that sph. star → ellipse! ....

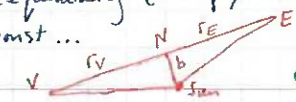
NB - gravitationally lensed images are laterally inverted.

Radar Echos

Photons: parametrise path with coordinate time:

$0 = \alpha c^2 - \frac{\left(\frac{dr}{dt}\right)^2}{\alpha} - r^2\left(\frac{d\phi}{dt}\right)^2$ but ϕ eq'n gives $r^2\frac{d\phi}{dt} = C$ so elim $\frac{d\phi}{dt}$. or, ...

...equivalently (I hope) let k and $h \to \infty$, but $W = \sqrt{\frac{h}{k}}$ does not - it is some const...



use point N to evaluate $W = \frac{b^2c^2}{1 - \frac{2\mu}{b}}$ $\Rightarrow \frac{dr}{dt} = c\left(1 - \frac{2\mu}{r}\right)\sqrt{1 - \frac{b^2\left(1 - \frac{2\mu}{r}\right)}{r^2\left(1 - \frac{2\mu}{b}\right)}}$

So can $\int dt = \int \frac{dr}{(\ )^{\frac{1}{2}}}$ to get time delay - must expand $\int$ to first order in μ.

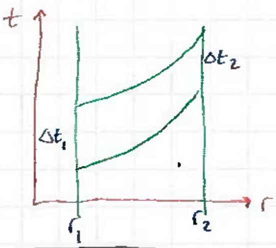
$$\longrightarrow \boxed{\Delta t_{rel} = \frac{2\mu}{c}\ln\left(\frac{2r_E}{b}\right) + \frac{\mu}{c}}$$

for $b \ll r_E$

for $E \to N$

$$\boxed{\Delta t_{rel\,tot} = \frac{4\mu}{c}\left[\ln\left(\frac{4r_E r_V}{b}\right) + 1\right]}$$

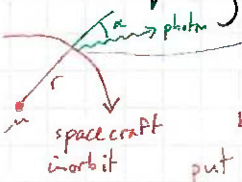
Gravitational Redshift



In Schwarzsch. metric, $\Delta t_1 = \Delta t_2$!

$$\Rightarrow \frac{\nu_2}{\nu_1} = \sqrt{\frac{\alpha_2}{\alpha_1}}$$

General Frequency Measurements



What freq does obs. at emission observe?

Inertial observer at rest measures

$\nu_{obs} = \nu_{em}.\gamma\left(1 + \frac{v\cos\alpha}{c}\right)$ of spacecraft

use $\Omega^2 = \frac{GM}{r^3}$

So coord vel $r\frac{d\phi}{dt} = \sqrt{\frac{GM}{r}}$ and $d\tau = \alpha dt$

put into S.R. doppler formula!

General Schwarzschild Geodesics

Use t, ϕ conserved quantities to get rid of t, ϕ dependence in eq'ns $p^2 = m^2$ and $p_r^2 = 0$:

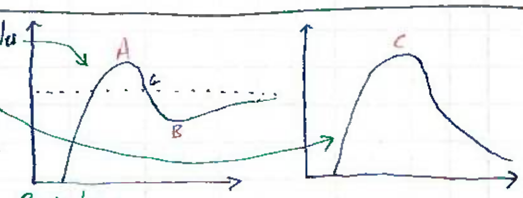
$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2\mu}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \quad \text{particles}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2\mu}{r}\right)\frac{L^2}{r^2} \quad \text{photons}$$

Effective Potentials

$V^2 = \left(1 - \frac{2\mu}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$ particles

$V^2 = \left(1 - \frac{2\mu}{r}\right)\frac{L^2}{r^2}$ photons



So: particle from ∞ with L only gets as far in as G....

← diff w.r.t. τ, get $F = ma$ - analogue of hyperbolic orbits

Photons + Particles: If $E^2 \gg V^2$ then for a given L, the impact parameter is small compared to smaller E.... then - goes straight past through $r \sim 2\mu$....

(Photons) For $b^2 = 27\mu^2$, can factorise, then get $\left(\frac{dx}{d\phi}\right)^2 = x^2 - x^3$ where $x = \frac{2}{3}(1 - 3u\mu)$

then let $z^2 = 1 - x$ to solve

get $\frac{1}{3} + \frac{2\mu}{r} = \left(\frac{Ae^{\phi} - 1}{Ae^{\phi} + 1}\right)^2$ then as $\phi \to \infty$, $r \to 3\mu$ - spirals in to circular orbit but never reaches.....

Photons

Only one, unstable orbit at C

$0 = \frac{d}{dr}(V^2) \Rightarrow \boxed{r = 3\mu}$

Capture if $E^2 < V^2_{max}$ ie at

now $b = \frac{L}{E}$ so must have

$E^2 < \frac{L^2}{27\mu^2}$ ie $b = 3\sqrt{3}\mu$

(particles - more complicated)

General orbit solution using W ie b, get no u term ∴ none or two

Particles:

Circular particle orbits at $r = A, B$:

get $r_{circ} = \frac{L^2}{2\mu}\left(1 \pm \sqrt{1 - \frac{12\mu^2}{L}}\right)$ so no sol'n for $L < 12\mu^2$

So subs $L = 12\mu^2$, get $r_{min} = 6\mu$!

General solution to $r(\phi)$ is: elliptic f'n:

$$\phi - \phi_0 = \int_{u_0}^{u} \frac{du}{\sqrt{au^3 - u^2 + \alpha u + \beta}}$$

where $a = 2\mu$, $\alpha = \frac{2\mu c^2}{h^2}$, $\beta = \frac{c^2}{h^2}(k^2 - 1)$

Orbits fall into distinct classes dep on pos'ns of roots in the cubic

① No real +ve roots → no bound orbit. $0 < r < \infty$
② One: aparentre at r_0, $0 \le r \le r_0$
③ Two: as above + a hyperbolic $r_1 < r < \infty$
④ Three: as above + something...

G.R. ③

Direct Infall

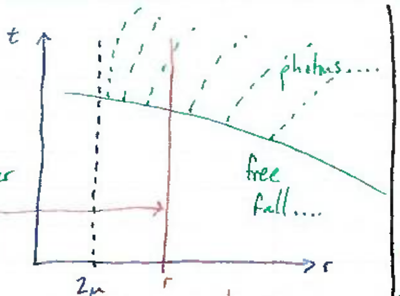
$c^2 = c^2 \alpha \dot{t}^2 - \frac{\dot{r}^2}{\alpha}$

use speed $= v_0$ (at ∞ to get $k = \gamma_0$)

→ then have coordinate speed $\dot{r}(t) \Rightarrow t(r)$...

stationary observer at radius r measures $t = \infty$ as $r = 2\mu$....

observer sees faller fall faster + faster → as obs. is really thrusting to stay at r.....



Black Holes (Schwarschild)

As collapse occurs, distant obs sees redshift → ∞ ....
Tidal forces cause great squeeze + stretch .... (geodesic deviation eq'ns...)

Schwarschild Solution with Cosmological Constant

$G_{ij} - \Lambda g_{ij} = -\frac{8\pi G}{c^4} T_{ij}$ then in vacuum, $R_{ij} = \Lambda g_{ij}$

but still sph sym, static: $ds^2 = e^A dt^2 - e^B dr^2 - r^2 d\Omega^2$

gives: $ds^2 = c^2 dt^2\left(1 - \frac{2\mu}{r} - \frac{\Lambda r^2}{3}\right) - \frac{dr^2}{\left(1 - \frac{2\mu}{r} - \frac{\Lambda r^2}{3}\right)} - r^2 d\Omega^2$

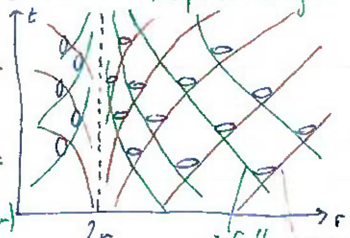
Geodesic eq'ns ⇒ $\frac{\Lambda r^2}{3}$ corresponds to a repulsive force: $u'' + u = \frac{\mu}{h^2} + 3\mu u^2 - \frac{\Lambda}{3h^2u^2}$ → additional advance of perihelion... → upper limit on Λ

let $\mu \to 0$, then get:
de-Sitter space! $ds^2 = c^2dt^2\left(1 - \frac{\Lambda r^2}{3}\right) - \frac{dr^2}{1 - \frac{\Lambda r^2}{3}} - r^2 d\Omega^2$

Singularities

ie at $r = 2\mu$.... but is it just due to coordinate choice? Helps to analyse for radial photons:

let's try to remove $r = 2\mu$ singularity with a coord tr'm that makes ingoing geodesics straight lines:

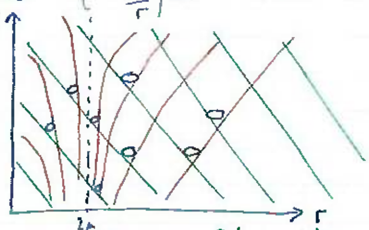


ie let $t \to \bar{t} = t + 2\mu \ln(r - 2\mu)$ or $2\mu - r$

then $\bar{t} = r + \text{const}$.... then:

$ds^2 = \alpha d\bar{t}^2 - \frac{4\mu}{r} d\bar{t}dr - \left(1 + \frac{2\mu}{r}\right)dr^2 - r^2 d\Omega^2$

Eddington-Finkelstein - no $r = 2\mu$ sing! so it's just a coordinate sing!



notice - nothing leaves $r = 2\mu$.... → Schw. radius.

Use advanced time: $v = c\bar{t} + r$:

$ds^2 = \alpha dv^2 - 2dvdr - r^2 d\Omega^2$ ← simpler form

or retarded: $w = ct^* + r$ → straightens outgoing...

$ds^2 = \alpha dw^2 + 2dwdr - r^2 d\Omega^2$

The Kerr Metric

Can solve Einstein's eq'ns directly or use null tetrads to go from Adv. E-Fink form to get form of Kerr metric:

$ds^2 = \left(1 - \frac{2\mu r}{\rho^2}\right)dv^2 - 2dvdr + \frac{2\mu r}{\rho^2}(2a\sin^2\theta)dvd\bar{\phi} + 2a\sin^2\theta dr d\bar{\phi}$
$- \rho^2 d\theta^2 - \left[(r^2+a^2)\sin^2\theta + \frac{2\mu r}{\rho^2}a^2\sin^4\theta\right]d\bar{\phi}^2$ $\quad \rho^2 = r^2 + a^2\cos^2\theta$

Properties of Kerr Solution

Parameters μ and a: let $a \to 0$, B-L ⇒ Schwarschild sol'n ∴ μ is $\frac{GM}{c^2}$ and a could be A.M.?

Kerr form: $\to M^4$ as $r \to \infty$!

The solution is stationary and axially symmetric

Also, the metric being unchanged on $t \to -t$, $\phi \to -\phi$ suggests spin.

And unchanged on $t \to -t$, $a \to -a$! So maybe a something to do with spin direction.

Also, when transform to rotating coord sys, let $\phi \to \phi - at$ which produces a $t\phi$ cross term in the metric........

→ Boyer-Lindquist form is closest to Schwarschild metric - use coord transformations:

$dv = cdt + dr = cdt + \left(\frac{2\mu + \Delta}{\Delta}\right)dr$

and $d\bar{\phi} = d\phi + \frac{a}{\Delta}dr$ where $\Delta = r^2 - 2\mu r + a^2$

then:

$ds^2 = \frac{\Delta}{\rho^2}(cdt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - acdt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$

Originally discovered in Kerr form:

let $c\bar{t} = v - r$, $x = r\sin\theta\cos\phi + a\sin\theta\sin\phi$
$y = r\sin\theta\sin\phi - a\sin\theta\cos\phi$
$z = r\cos\theta$

then:

$ds^2 = [M^4] - \frac{2\mu r^3}{r^4 + a^2z^2}\left(c^2d\bar{t} + \frac{r}{a^2+r^2}(xdx + ydy) + \frac{a}{a^2+r^2}(ydx - xdy) + \frac{z}{r}dz\right)^2$

Minkowski space + a correction!

Kerr Singularities

$R^{ijkl}R_{ijkl} \to \infty$ as $\rho \to 0$

⇒ $r \to$ and $\cos\theta \to$, ⇒ $x^2 + y^2 = a^2$, $z = 0$ from ...

ie there is a ring singularity!!!

Infinite Redshift Surfaces (Stationary limit surfaces)

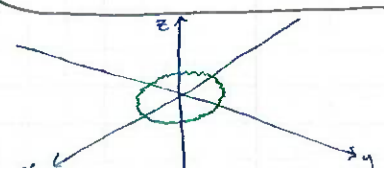
let $g_{00} \to 0$, get (from B-L), $r = \mu \pm \sqrt{\mu^2 - a^2\cos^2\theta} = r_{S\pm}$

Event horizon (for photons....) let $g_{11} \to \infty$ or $g^{11} \to 0$

then, (B-L again) get $\Delta = 0 \Rightarrow r = \mu \pm \sqrt{\mu^2 - a^2} = r_{H\pm}$

They coincide as $a \to 0$! Inside the ring, the metric has $r < 0$ ∴ no more horizons.

Just inside ring ∃ closed timelike curves as $g_{\phi\phi} < 0$ ie $a^2\left(1 + \frac{2\mu}{r}\right.$



G.R. ④

Kerr Geodesics

No radial null geodesics anymore – frame dragging!

Try $\theta = $ const plane?: using B-L:

$$A = \frac{\Delta}{\rho^2}(c\dot{t} - a\sin^2\theta\,\dot{\phi}) + \frac{a^2\sin^2\theta}{\rho^2}\left[(r^2+a^2)\dot{\phi} - a\dot{t}\right] \quad (t)$$

$$B = \frac{a\Delta\sin^2\theta}{\rho^2}(c\dot{t} - a\sin^2\theta\,\dot{\phi}) + \frac{(r^2+a^2)\sin^2\theta}{\rho^2}\left[(r^2+a^2) - a\dot{t}\right] \quad (\phi)$$

$$0 = \frac{\Delta}{\rho^2}(c\dot{t} - a\sin^2\theta\,\dot{\phi})^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)\dot{\phi} - ac\dot{t}\right]^2 - \frac{\rho^2\dot{r}^2}{\Delta} \quad (r)$$

↑ null

θ eq'n gives a 4th eq'n but $\dot{\theta} = 0$ so can use to relate A to B:

$\rightarrow (B + aA\sin^2\theta)(B - aA\sin^2\theta) = 0$

then $\rightarrow \dot{t} = \frac{(r^2+a^2)A}{\Delta}, \; \dot{r} = \pm A, \; \dot{\phi} = \frac{aA}{\Delta}$

wow!!

now r is an affine parameter! $r = \pm A\lambda + C$

so let's use it:

$$\frac{dt}{dr} = \frac{r^2+a^2}{\Delta}, \qquad \frac{d\phi}{dr} = \frac{a}{\Delta}$$

can integrate to give:

$$ct = r + \mu\ln\left(|r - r_{H+}||r - r_{H-}|\right) + \frac{\mu^2}{(\mu^2-a^2)^{1/2}}\ln\left(\frac{|r-r_{H+}|}{|r-r_{H-}|}\right) + \text{const}$$

$$\phi = \frac{a}{2(\mu^2-a^2)^{1/2}}\ln\left(\frac{r-r_{H+}}{r-r_{H-}}\right) + \text{const.}$$

← using $\dot{r} = +A$ – can just let $t \to -t$, $\phi \to -\phi$ for $\dot{r} = -A$

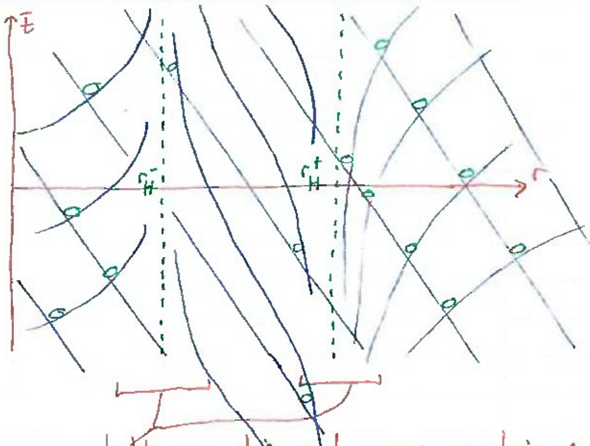
can straighten the ingoing geodesics with coord tr'm:

$$\frac{c\,d\bar{t}}{dr} = \frac{c\,dt}{dr} + \frac{2\mu r}{\Delta}$$

and

$$\frac{d\bar{\phi}}{dr} = \frac{d\phi}{dr} + \frac{a}{\Delta}$$

→ then:



light cones tip and narrow near horizons

Consider B-L metric with $dr = d\theta = 0$

- get $\frac{\Delta}{\rho^2}(c\,dt - a\sin^2\theta\,d\phi)^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - ac\,dt\right]^2 = 0$

then can get $\frac{d\phi}{dt} = \frac{c(a\sin\theta \pm \sqrt{\Delta})}{(r^2+a^2)\sin\theta \pm a\sqrt{\Delta}\sin^2\theta}$ ← not geodesics... just writ. cond...

take + sign: then $\frac{d\phi}{dt} > 0$ and photons go with the flow.

take – sign, then $\frac{d\phi}{dt} < 0$ outside r_S^+, $\frac{d\phi}{dt} = 0$ on r_S^+

so photon appears to stand still on (∞ z surface)

as go through r_S^+, $\frac{d\phi}{dt}$ is forced → +ve

So photons must go with flow.

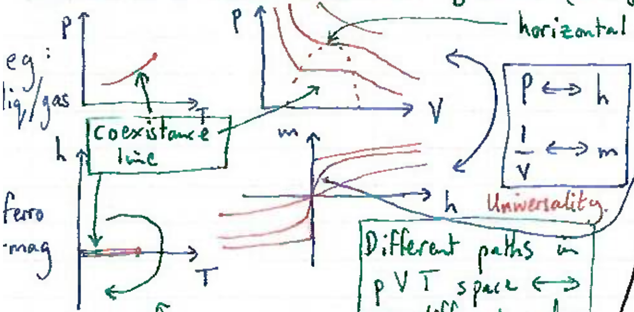
[If $a^2 > \mu^2$, get no event horizon but still have ring singular ∴ can see it! (Naked Singularity) but if A.M. > Grav, maybe whole thing flies apart....]

Ph. Tr ①

Z for finite no. of particles is always analytic but ph tr $\Leftrightarrow$ Z sing! So ph tr only in thermod. limit

The Basics

So must characterise the singularities (in $\log Z$).



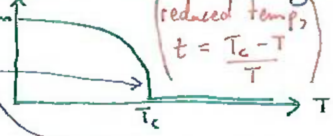
- can do this $\because$ no fund. diff between phases.

1st order transition: order parameter disc.
2nd order " : order parameter cont.

At the critical point:

Same crit. expts $\Leftrightarrow$ same universality class

Order param.: $m = \frac{1}{V} \lim_{h \to 0} M(h, T)$

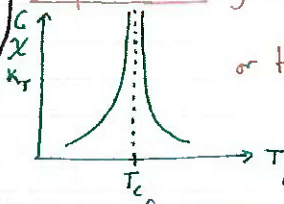


(reduced temp, $t = \frac{T_c - T}{T}$)

Here, $m \propto |t|^{\beta}$ and here, (on $T = T_c$), $m \propto h^{\frac{1}{\delta}}$ eq'n of state!

Response f'ns: eg compressibility $K_T = \frac{1}{V}\frac{\partial V}{\partial P}$, suscep. $\chi_{\pm}\Big|_{h=0} = \frac{\partial m}{\partial h}\Big|_{h=0}$

or Heat capacity $C_{\pm} = T \frac{\partial S}{\partial T}\Big|_V$



and: $\chi_{\pm}\Big|_{h=0} \propto |t|^{-\gamma_{\pm}}$, $C_{\pm}\Big|_{h=0} \propto |t|^{\alpha_{\pm}}$

Divergence of response functions $\Longleftrightarrow$ fluctuations are correlated over long distances

$Z = \sum e^{-\beta(H_0 - hM)}$ where $M = \int d^3\underline{x}\, m(\underline{x})$ and $\langle M \rangle = \frac{d \ln Z}{d(\beta h)}$

$\Rightarrow \chi = \frac{\beta}{V}\left(\langle M^2 \rangle - \langle M \rangle^2\right)$
$= \frac{\beta}{V} \int d^3x\, d^3x' \left(\underbrace{\langle m(\underline{x}) m(\underline{x}') \rangle}_{G(x-x')} - \underbrace{\langle m(x) \rangle \langle m(x') \rangle}_{m^2}\right)$

Connected correlation function $\langle m(\underline{x})\, m(0) \rangle_c$

Typically $G_c(x, 0) \propto e^{-\frac{x}{\xi}}$ and $kT\chi < g\xi^3$ where g is $G_c(x)$ for one cell.

So if χ diverges so does ξ: $\xi_{\pm}\Big|_{h=0} \propto |t|^{-\nu_{\pm}}$ - coarse grains so no need for full microscopic description.

Ginzburg - Landau Hamiltonian

use n component order parameter in d dimensions $\underline{m}(\underline{x})$ where $\underline{m}(\underline{x})$ is averaged over a lattice cell ie no Fourier components for $\Lambda > \frac{1}{a}$.

- $d = 1$: liq-gas, binary mixtures.
- $d = 2$: superfluids, superconductors
- $d = 3$: 3-d isotropic magnets

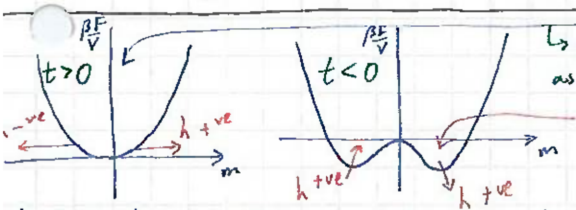
Locality (so use gradient expansions) and rotational symmetry (so use dot products) in $\underline{m}$ space and trans/rot sym. in x-space mean that the Ham. is:

$H[\underline{m}] = H[R_n \underline{m}]$

$H[\underline{m}(\underline{x})] = H[\underline{m}(\underline{R}\underline{x})]$

$$\beta H = \int d^d\underline{x} \left[\frac{t}{2}m^2 + u m^4 \ldots + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m^2)^2 + \frac{N}{2} m^2 (\nabla m)^2 \ldots - \underline{h}\cdot\underline{m}\right]$$

params are f'ns of microscopic (interactions) params and also of temp, press etc coz $\exists$ entropy involved with coarse graining!!



↳ can just keep low powers as near $T = T_c$ m is small

spontaneous symmetry breaking (in m-space)

to first order in t $\exists$ a map between magnets and G-L ....

Mean Field Theory

$Z[\underline{h}] = \int \mathcal{D}\underline{m}(x)\, e^{-\beta H[\underline{m}, \underline{h}]}$ for $K > 0$, $\underline{m}$ that minimises H is indep of pos'n

So let $\underline{m}(\underline{x}) = \bar{m}\, \underline{e}_h$ then $\int d^d\underline{x} = V$ and must minimise free energy density: $\beta\left(\frac{F}{V}\right) = \frac{t}{2}m^2 + u m^4 - h \cdot m$

so $\min(\beta H) = \beta F$ and:

$Z = e^{-\beta H_{min}} \int \mathcal{D}m\, e^{-\beta\underbrace{(H - H_{min})}_{H_{fluct.}}}$ k-space, modes decouple at Gaussian order

$F = \ln Z = F_{M.F.} + \delta H_{fluct} \to 0$ dep. on dimension.

M.F. critical exponents:

Magnetisation: $\frac{\partial(\frac{\beta F}{V})}{\partial m} = 0 \Rightarrow \bar{m} = \left(\frac{-t}{4u}\right)^{1/2}$ t -ve

So $\beta = \frac{1}{2}$

Heat Capacity: $C = -T\frac{\partial^2 F}{\partial T^2} = \frac{k}{8u}$ for $t < 0$ so $\alpha = 0$ there is only a jump

Susceptability: $\frac{1}{\chi} = \frac{\partial h}{\partial m}\Big|_{h=0} = t + 12um^2$ so $\gamma = 1$

(χ_+, χ_- dep on microscop. but their ratio is univ.)

Equation of state: for $t = 0$ (on critical isotherm) $\Rightarrow \delta = 3$

Functional Integration

Simplest case: (Discrete case) $Z_1 = \int_{-\infty}^{\infty} d\phi\, e^{-\frac{\phi^2}{2G} + h\phi} = \sqrt{2\pi G}\, e^{\frac{Gh^2}{2}}$

$\langle \phi^n \rangle_c = \frac{\partial^n}{\partial h^n}(\ln Z_1)\Big|_{h=0}$ and $\langle \phi^n \rangle = \frac{1}{Z}\frac{\partial^n Z}{\partial h^n}\Big|_{h=0}$

For N variables:

$Z_N = \int_{-\infty}^{\infty} d\phi_1 \cdots d\phi_N\, e^{-\frac{1}{2}\underline{\phi}^T \underline{G}^{-1} \underline{\phi} + \underline{h}^T \underline{\phi}}$ diagonalise $\underline{G}^{-1}$ to produce a product of Z_1 integrals....

diagonalise + complete sq.: let $\underline{\phi}' = U\underline{\phi} - \tilde{G}U\underline{h}$, $\tilde{G}^{-1} = UG^{-1}U^{-1}$

then $Z_N = \int_{-\infty}^{\infty} d\phi_1' \cdots d\phi_N'\, e^{-\frac{1}{2}\underline{\phi}'^T \tilde{\underline{G}}^{-1} \underline{\phi}' + \frac{1}{2}\underline{h}^T \underline{G}\underline{h}} = \det(2\pi\underline{G})^{\frac{1}{2}}\, e^{\frac{1}{2}\underline{h}^T\underline{G}\underline{h}}$

equivalently (easier to remember) let $\underline{\phi} \to \underline{\phi} - \underline{G}\underline{h}$

then: $Z_N = e^{\frac{1}{2}\underline{h}^T\underline{G}\underline{h}} \int d\phi_1 \cdots d\phi_N\, e^{-\frac{1}{2}\underline{\phi}^T \underline{G}^{-1} \underline{\phi}}$

now make the unitary (diagonalising) transformation

but anyway, $Z_N(\underline{h}) = Z_N(\underline{0})\, e^{\frac{1}{2}\underline{h}^T\underline{G}\underline{h}}$ as before.

Ph. Tr. ②

Connected part of Z_N is: $\langle \phi_i \cdots \phi_j \rangle_c = \frac{\partial}{\partial h_i} \cdots \frac{\partial}{\partial h_j} \Big|_{h=0} \ln Z_N$

For Gaussians $\langle e^A \rangle = e^{\langle A \rangle_c + \frac{\langle A^2 \rangle_c}{2}}$ useful

$= \frac{\partial}{\partial h_i} \cdots \frac{\partial}{\partial h_j} \Big|_{h=0} \ln \left(\langle e^{h^T \phi} \rangle \right)$ ← Gaussian average

So $\langle \phi \rangle_c = G h$ and $\langle \phi_i \phi_j \rangle_c = G(ij)$, $= 0$ for higher orders.

Functional Integration

(Continuous limit)

$G_{ij} \longrightarrow$ operator....

get:

$$Z_\infty = \int D\phi(x)\, e^{-\frac{1}{2}\int d^dx\, d^dx'\, \phi(x)\, G^{-1}(x-x')\, \phi(x') + \int d^dx\, h(x)\, \phi(x)}$$

$$= \text{Factor} \times (\det G)^{1/2}\, e^{\frac{1}{2}\int d^dx\, d^dx'\, h(x)\, G(x-x')\, h(x')}$$

then:

$$\langle \phi(x) \rangle_c = \int d^dx\, G(x-x')\, h(x') \quad \text{and} \quad \langle \phi(x)\phi(x') \rangle_c = G(x-x')$$

LOOK!! if apply $\frac{\delta}{\delta h(x)}$ to Z_∞, bring down a $\phi(x_1)$ but still have all the consts. of proportionality! What we have is $\langle \phi(x_1) \rangle$ - all the diagrams. If then normalise - ie divide by $Z_\infty(0)$, (the vacuum bubs)! then get the connected part only!! ie the correct vacuum......!

if have G-L Ham, $\nabla\phi^2 + \frac{\phi^2}{\xi^2}$, integrate by parts to get:

$G^{-1}(x, x') = \delta^d(x-x')\left(-\nabla^2 + \frac{1}{\xi^2}\right)$ - can invert using definition of G^{-1}:

$$\int d^dx'\, G^{-1}(x, x')\, G(x', x'') = \delta^d(x - x'') \quad \to \text{Green function!}$$

Goldstone Modes

For $T > T_c$, no prefer magnetic moment dir'n. For $T < T_c$, $m \to \bar{m}\, e$, and this rotational symmetry is spontaneously broken. Goldstone says that this $\Rightarrow \exists$ massless (low energy) excitations of the field - in this case they are spin-waves. For a solid they are phonons.

Eg X-Y model in d-dimensions:

$m = \bar{m}(\cos\theta, \sin\theta)$: $\beta H = \beta H_0 + \frac{\bar{K}}{2}\int d^dx\, (\nabla\theta)^2$ ← propagator (Laplacian)

Fourier transform $\theta(x)$ then get spin wave modes

then: Correlation f'n is:

$$\langle \theta(x)\, \theta(x') \rangle = G(x, x') = -\frac{C_d(x-x')}{\bar{K}} \quad \text{where} \quad \nabla^2 C_d(x) = \delta^d(x)$$

C_d is Coulomb potential for δ-f'n charge dist'n. Use Gauss's law to find:

$$C_d = \frac{x^{2-d}}{(2-d)S_d} \qquad \left(S_d = \int_{d\text{-dim sphere}} d\Omega = \frac{2\pi^{d/2}}{\left(\frac{d}{2}-1\right)!} \right)$$

Can see already: LRO destroyed for $d \leq 2$ ie if phase fluctuations are very correlated, because bounded by 2π, they are not correlated.....?

Fluctuation Corrections to Mean Field Theory

Set $m(x) = [\bar{m} + \phi_L(x)]\, \hat{e}_1 + \sum_{\alpha=2}^{n} \phi_t^\alpha(x)\, \hat{e}_\alpha$ (ϕ_L small, $\hat{e}_1$ unit vector, ϕ_t^α small, $\hat{e}_\alpha$ unit vector)

Subst into G-L Hamiltonian, keeping quad. in ϕ:

get:

$$\beta H = L^d\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \int d^dx \left(\frac{K}{2}(\nabla\phi_L)^2 + \frac{t + 12u\bar{m}^2}{2}\phi_L^2\right)$$

$$+ \int d^dx \left(\frac{K}{2}(\nabla\phi_t)^2 + \frac{t + 4u\bar{m}^2}{2}\phi_t^2\right) + O(\phi^3)$$

This is diagonalised in Fourier-space, giving:

$$\beta H = \sum_q -\frac{K}{2}\left(q^2 + \frac{1}{\xi_L^2}\right)|\phi_L(q)|^2 - \frac{K}{2}\left(q^2 + \frac{1}{\xi_t^2}\right)|\phi_t(q)|^2 \quad (+\beta H_0)$$

where $\xi_L^2 = \frac{K}{t + 12u\bar{m}^2}$ and $\xi_t^2 = \frac{K}{t + 4u\bar{m}^2}$

$\frac{K}{\xi^2}$ act as spring constants ie energy $= \frac{1}{2}kx^2$

use M.F. theory to get $\bar{m}(t)$ then find:

for $t > 0$ $\frac{K}{\xi_L^2} = \frac{K}{\xi_t^2} = t$, $t < 0$ $\frac{K}{\xi_L^2} = -2t$, $\frac{K}{\xi_t^2} = 0$

ie no prefered direction for above ph. tr. point; no restoring force → Goldstone modes.

Do the Gaussian Functional Integrals, get:

$$\langle \phi_\alpha(q)\, \phi_\beta(q') \rangle_c = \delta_{\alpha\beta}\, \delta_{q,-q'}\, G(q)$$

where can read off G:

$$G^{-1}(q) = K\left(q^2 + \frac{1}{\xi_\alpha^2}\right)$$

Lorentzian in Fourier Space.

So what does the correlation look like in real space? need to do the (Inverse) Fourier Transform:

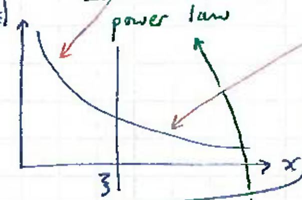
$$G_{\alpha\beta}(x) = \int \frac{d^dq}{(2\pi)^d} \frac{e^{iq\cdot x}\, \delta_{\alpha\beta}}{\left(q^2 + \frac{1}{\xi^2}\right) \cdot K} = \langle \phi_\alpha(x), \phi_\beta(0) \rangle_c$$

- tricky in d-dimensions. (Feynman parameters)

Assymptotically,

for $x \ll \xi$, $G(x) = \frac{x^{2-d}}{(2-d)S_d}$ (power law), for $x \gg \xi$ $G(x) \propto \frac{e^{-x/\xi}}{x^{\frac{d-1}{2}}}$ (exp. in 1-d....)

So:



close to the transition (t small) get $\xi_L = \left(\frac{K}{t}\right)^{1/2}$ above and $= \left(-\frac{K}{2t}\right)^{1/2}$ below $\Big\}$ $\xi \propto |t|^{-\frac{1}{2}}$

So at the transition, $t = 0$, $\xi_L = \infty$ so

So $\nu_+ = \nu_- = \frac{1}{2}$

Susceptability: $\int G_L^{con}(x)\, d^dx = \chi_L \propto \int_0^{\xi_L} \frac{d^dx}{x^{d-2}} \propto \xi_L^2 \sim t^{-1} \therefore \gamma = 1$!

for $t < 0$, $\xi_t = \infty$ so $G_t^{con} \propto \int_0^L \frac{d^dx}{x^{d-2}} \propto L^2$ (system size)

big - very susceptable - no energy cost to rotate m - Goldstone Bosons...

Want Z now:

- Can do Gaussian integral:

$$\int D\phi_L\, D\phi_t\, e^{-\beta H_0 - H(\phi)} = e^{-\beta H_0}(\det G)^{1/2}$$

but then can use $\ln(\det G) = -\mathrm{Tr}(\ln G)$!

$$e^{-\beta H_0 + \frac{1}{2}\ln\det G} \to \frac{1}{2}\int(dq)\, \ln\left[K\left(q^2 + \xi_L^{-2}\right)\right] + \left(\frac{n-1}{2}\right)\int(dq)\left[K\left(q^2 + \xi_t^{-2}\right)\right]$$

ln of this!!!

now use → and $C = -\frac{\partial^2 f}{\partial t^2}$

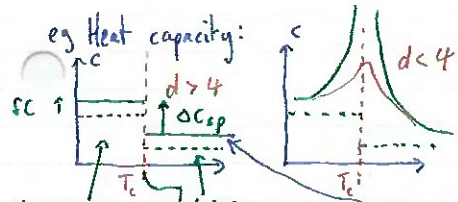
- get: $C_{singular} \propto \int \frac{(dq)}{(K\, q^2 + t)^2}$ diverges at large q for $d > 4$

$\delta C \sim \frac{1}{K^2} \times \begin{cases} a^{4-d} & d > 4 \\ \xi^{4-d} & d < 4 \end{cases}$ 4 is the upper-critical dimension.

Ph Tr ③

Upper-critical dimension

eg Heat capacity:



$\frac{n}{2}\int(dq)\frac{1}{(Kq^2+t)^2}$ | $\frac{1}{8u} + 2\int(dq)\frac{1}{(Kq^2-2t)^2}$

integral converges for $d<4$ but UV diverges for $d>4$ ∴ dep on cutoff $\frac{1}{a}$

rescale q by ξ to make dimensionless

Then: $\propto \xi^{4-d}$ - do same with a for $d>4$ so:

$$\delta C \sim \frac{1}{K^2}\times\left[\begin{matrix} a^{4-d} & \text{for } d>4 \\ \xi^{4-d} & \text{for } d<4\end{matrix}\right]$$

- diverges for $d<4$ (IR div)
- would diverge for HEP QFT, $d>4$... (no cutoff.)

$d>4$, const is added by fluctuations to both sides of T_c

$d<4$ $\delta C \gg C$ ∴ MF no longer valid.

know $\xi \propto t^{-1/2} \to$ get α

- get same result of any other quantity eg mag, susc.....

Summary:

Ginzburg Criterion

Expts done on some systems in $d=3$ show MF exact, other's MF approx but $3 < $ u.c.d.! ∴ MF should only be approx in all cases.....

Can estimate when fluct becomes important by saying when is $\Delta C_{\text{saddle point}} \simeq \delta C$, correction. → well, ΔC_{sp}

Scaling and Homogeneity

before, eg free energy $f_{MF} \propto \frac{-t^2}{u}$ for $h=0, t<0$ or $\frac{h^{4/3}}{u^{1/3}}$ for $h\neq 0, t=0$

If let f have homogeneous form ($f(x) = b^k f(bx)$) then reproduce the behaviour here

ie let $f(t,h) = t^2 g_f\left(\frac{h}{t^\Delta}\right)$ using, find $\Delta = 3/2$ works ↑ gap exponent.

For homogeneous functions:

$$\lim_{x\to 0} g_f(x) = -\frac{1}{u}$$

$$\lim_{x\to+\infty} g_f(x) = \frac{x^{4/3}}{u^{1/3}}$$

Assumption of Homogeneity:

- that free energy etc can be written as homogeneous even when beyond saddle point approx ie any field config...

ie $f_{\text{singular form}}(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^\Delta}\right)$

Derive thermod. quantities from f, get relations between exponents and α, Δ by requiring same behaviour. ↳ eg $m(t,h) = \frac{\partial f}{\partial h} = t^{2-\alpha-\Delta} g_m\left(\frac{h}{t^\Delta}\right)$ $\Rightarrow \beta = 2-\alpha-\Delta$

- Singular parts of all critical quantities are hom.
- Same gap exponent Δ for each ∫. (Universality)
- Only 2 indep ones, (α, Δ).

Hyperscaling

To involve the correlation length, replace assum. of homog. by:

1. Correlation length is homogeneous: $\xi(t,h) \sim t^{-\nu} g_\xi\left(\frac{h}{t^\Delta}\right)$ for $t=0$, ξ diverges as $\frac{1}{h^{\nu/\Delta}}$
2. As $t\to 0$, ξ is the sole controller of thermod. quantities

$\Rightarrow \ln Z = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a$ as $\ln Z$ dimensionless and extensive ($\propto L^d$)

↑ non singular, f(params)

$f_{sing} \sim \frac{\ln Z}{L^d} \sim \xi^{-d}$ then condition 1. $\Rightarrow$ $f_{sing}(t,h) \sim t^{d\nu} g_f\left(\frac{t}{h^\Delta}\right)$ homogeneity recovered from ξ.

$\Rightarrow d\nu = 2-\alpha$ ← Hyperscaling Relation (coz involves d).

turns out to be valid in low dimensions but breaks down for $d>4$.

Correlation Functions also are homogeneous at $t=0$

$$G_{\text{critical}}(\lambda \underline{x}) = \lambda^p G_{\text{critical}}(\underline{x})$$

ie self similar - same apart from change in contrast, λ^p.

Tricky to build this into Ham. - (ie just add dilation symmetry to constraint list)

Renormalisation Group

Conceptual Approach

Integrate out fast d.o.f. $\underline{m}(x) \to \bar{\underline{m}}(x)$

Restore resolution by rescaling $\underline{x} \to x' = \frac{x}{b}$ where average over box (b^d)

Restore contrast by rescaling $\bar{\underline{m}}(x) \to m'(x') = \frac{1}{\zeta}\bar{\underline{m}}(x')$

If at critical point, self sim. ⇒ no change in Hamiltonian parameters

Critical point means $t=h=0$ ∴ $t'=h'=0$.

But if not at critical point, we are taken further away as $\xi_{new} = \frac{\xi}{b}$!

now: $t' \simeq A(b)t + B(b)h$
$h' \simeq C(b)t + D(b)h$ } assume analytic

to first order (no const. term coz) $B=C=0$ to prevent spontaneous symmetry breaking and commutativity (?) ⇒

"semi-group" property!

$$\begin{Bmatrix} A = b^{y_t} \\ D = b^{y_h}\end{Bmatrix} \quad \begin{matrix} y_t > 0 \\ y_h \end{matrix}$$

y_t, y_h are related to the critical exponents:

eg free energy: $\int Dm\, e^{-\beta H[m]}$ must $= \int Dm'\, e^{-\beta H[m']}$ ie $Z = Z'$ So $f = \frac{\ln Z}{V}$ only changes through V: $f(t,h) = \frac{f(t',h')}{b^d}$

ie $f(t,h) = b^{-d} f(b^{y_t}t, b^{y_h}h)$

but this is the definition of a homogeneous function.

For a given b, say $b = t^{-1/y_t}$ then $\Delta = \frac{y_h}{y_t}$, $2-\alpha = \frac{d}{y_t}$

can get all critical exponents from

eg correlation length:

$$\xi(t,h) = b\,\xi(b^{y_t}t, b^{y_h}h) = t^{-1/y_t}\xi\left(1, \frac{h}{t^{y_h/y_t}}\right) \propto t^{-\nu}$$

so $\nu = \frac{1}{y_t}$ and $2-\alpha = \nu d$ Hyperscaling Identity.

magnetisation:

$$m(t,h) = \frac{1}{V}\frac{\partial \ln Z}{\partial h} = \frac{1}{b^d V'}\frac{\partial \ln Z'(t',h')}{b^{-y_h}\partial h}$$

ie $m(t,h) = b^{y_h-d} m(b^{y_t}t, b^{y_h}h)$

- for conjugate variables eg $\underline{m}\cdot\underline{h}$ always: $y_m + y_h = d$.....

Ph. Tr. (4)

Renormalisation Group

Formal Approach

R.G. sends: $\beta H[\underline{m}(x)]$

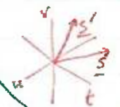
$= \int d^dx \left[\frac{t}{2} \underline{m}^2 + u\underline{m}^4 + v\underline{m}^6 + \dots + \frac{K}{2}(\nabla \underline{m})^2 + \dots \right]$

to $\beta H'[\underline{m}'(x')]$

$= \int d^dx' \left[\frac{t'}{2} \underline{m}'^2 + u' \underline{m}'^4 + \dots \right]$

where $\underline{m}'(x') = \frac{1}{\zeta b^d} \int d^dy \, m(y)$ (cell, vol b^d centred on $\underline{x} = b\underline{x}'$)

This is a mapping in parameter space:



$\underline{S} \to \underline{S}' = R_b(\underline{S})$

(not necessarily linear...)

So fixed points ≡ critical pts

ie $R_b S^* = S^*$

but $\xi(\underline{S}) = b\,\xi(R_b \underline{S})$

$\Rightarrow \xi = 0$ or ∞ if $S = S^*$!

but $\xi = 0 \Rightarrow T = \infty$ or zero (think!)

$\therefore \xi = \infty$ is critical point.

• Near S^*, can linearise R_b: $\underline{S}^* + \delta\underline{S} \to \underline{S}^* + R_b^{\text{lin}} \delta \underline{S}$

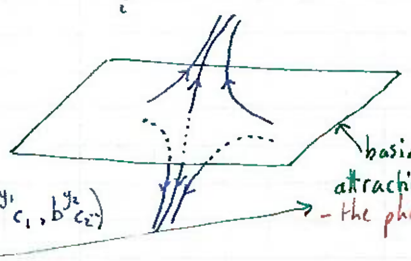
$R_{b'}^L R_b^L O_i = R_{b'b}^L O_i$ (group property)

$\lambda(b')_i \lambda(b)_i O_i$ = $\lambda(b'b)_i O_i$ ← ith eigenvector of R_b^L ← scaling directions

eigenvalues.... then if let $\lambda(1)_i = 1$, we get $\lambda(b)_i = b^{y_i}$ ← anomalous dimensions

$\beta H \approx \beta H^* + \sum c_i O_i$, then after applying R.G., get: $\beta H' = \beta H^* + \sum_i c_i b^{y_i} O_i$ — Near to fixed point.

Then if: $y_i > 0$, $c_i \uparrow$, O_i is relevant
$y_i < 0$, $c_i \downarrow$, O_i is irrelevant
$y_i = 0$, $c_i \cdot$, O_i is marginal – need higher orders....



near S^*, $\xi(\,,\,\dots) = b\,\xi(b^{y_1}c_1, b^{y_2}c_2)$

so on basin, $\xi = \infty$

(otherwise – would change) As $b \to \infty$, all irrelevant operators → zero....

Let y_1 have dim > dim(y_2) > .....

then: $\xi(c_1, c_2 \dots) = c_1^{-\frac{1}{y_1}} f\left(\frac{g_2}{g_1^{y_2/y_1}}, \dots\right)$ → so $\nu_1 = \frac{1}{y_1}$ and $\Delta_\alpha = \frac{y_\alpha}{y_1}$ – set of gap expts

Usually dim of basin

For fixed pts describing phases have dim = dim of potential ?!!

For pts describing critical pts, have basins have < potential ?

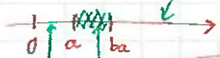
Universality is explained – microscopic details make up the space of irrelevant operators .... Generally, for fixed points describing 2nd order ph. tr. there are two relevant params: t and h..... (conj. field!...)

RG Applied to Gaussian Model

$Z_{[h]} = \int \mathcal{D}\underline{m}(x)\, e^{-\int d^dx \left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 - h \cdot m\right]}$ ↘ $t \geq 0$ as no m^4 term!

R.G. treatment:

① Coarse grain:

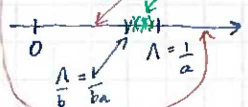


leave these...

no integral here anyway

integrate over these d.o.f.s

Same as:



: let $\underline{m}(q)$ be split into $m_>$, $m_<$

then: $Z = \int \mathcal{D}m_< \int \mathcal{D}m_> e^{-\beta H[m_<, m_>]}$ modes decouple at Gaussian order...

$= Z_> \int \mathcal{D}m_<(q)\, e^{-\int_0^{\Lambda/b}(dq)\left(\frac{t + Kq^2}{2}\right)|m_<(q)|^2 + h \cdot m_<(0)}$

$Z_> = e^{-\left(\frac{nV}{2}\right)\int_{\Lambda/b}^{\Lambda}(dq)\ln(t + Kq^2)}$

② Rescale: let $q \to q' = bq$ (ie $x' = x/b$)

③ Renormalise: let $\underline{m}'(x') = \frac{m_<(x')}{\zeta}$

or in q space, $z \to \zeta$, then:

$Z = Z_> \int \mathcal{D}m'(q')\, e^{-\beta H'[m'(q')]}$ where:

$\beta H' = \int_0^\Lambda (dq')\, b^{-d} z^2 \left(\frac{t}{2} + \frac{Kb^{-2}}{2}q'^2\right)|m'(q')|^2 - z\,h \cdot m'(0)$

So: the transformation is:
$K' = b^{-d-2} z^2 K$
$t' = b^{-d} z^2 t$
$h' = z h$

The fixed point we know about (2nd order ph. tr.) is $t = h = 0$ but for this to be fixed $\forall K$ requires $b^{-d-2} = z^{-2}$ ie $z = b^{1 + d/2}$

then: $t' = b^2 t \Rightarrow y_t = 2$, > 0 so relevant
$h' = b^{1+d/2} h \Rightarrow y_h = 1 + \frac{d}{2}$ also relevant

There is another at $t = t'$, $h = 0$ ∴ $z = b^{+d/2}$, $K \to b^{-2} K$ (–ve!)

– this is the high temperature phase.

Free energy: $f_{sing}(t, h) = b^{-d} f_{sing}(b^2 t, b^{1+d/2} h)$

let $b^2 t = 1$ then: $= t^{d/2} g_f\left(\frac{h}{t^{\frac{1}{2}+\frac{d}{4}}}\right)$

So $2 - \alpha = \frac{d}{2}$, $\Delta = \frac{1}{2} + \frac{d}{4}$, $\nu = \frac{1}{2}$ ie $\frac{1}{y_t}$ c.f. exact solution

At the fixed point $t = h = 0$, $z = b^{1+d/2}$, system is scale invariant.

By dimensional analysis? $(\beta H)^* = \int d^dx' \frac{K}{2} b^{d-2} \zeta^2 (\nabla m')^2$ so $\zeta = b^{1-d/2}$

Also, for small perturbations away from the fixed point,

$(\beta H)^* + u_p \int d^dx\, |m|^p \to (\beta H)^* + u_p b^d \zeta^p \int d^dx' \, |m'|^p$

So $u_p \to u_p' = b^{p - d(\frac{p}{2} - 1)} u_p$ ie $y_p = p - d\left(\frac{p}{2} - 1\right)$

$\Rightarrow y_1(y) = 1 + \frac{d}{2}$ and $y_2(y) = 2$ see above!!!

also $y_4 = 4 - d$ think! $y_6 = 6 - 2d$ think!

Ph. Tr. ⑤

Wilson's Perturbative Approach

Now: include $U = u\int d^dx\, \underline{m}^4$ in βH. In Fourier space,

$$U = u\int (dq_1)\cdots(dq_4)\left[\underline{m}(q_1)\cdot\underline{m}(q_2)\right]\left[\underline{m}(q_3)\cdot\underline{m}(q_4)\right] \times \delta^d(q_1+q_2+q_3+q_4)$$

modes don't decouple

then: ① coarse grain:

$$Z = \int Dm_> Dm_< e^{-\beta H_0[m_>]-\beta H_0[m_<]-U[m_<,m_>]}$$

$$= \int Dm_< e^{-\beta H_0[m_<]} \int Dm_> e^{-\beta H_0[m_>]-U}$$

$$\langle e^{-U}\rangle_{m_>} \times Z_0^>$$

$$= \int Dm_< e^{-\beta H_0[m_<] + \ln\langle e^{-U}\rangle_{m_>} + \ln Z_0^>}$$

expand in perturbation series – get cumulants !!!!!!

only keep first order term

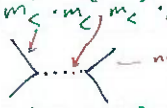
$-\langle U\rangle_c \;(= -\langle U\rangle)$

(still averaging over $m_>$)

$\langle U\rangle_{m_>}$ has terms like:

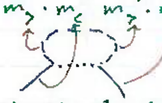
$C_1 = \langle m_<^1 \cdot m_<^2\, m_<^3 \cdot m_<^4\rangle_{m_>}$

ie



$C_2 = \langle m_>^1 \cdot m_>^2\, m_<^3 \cdot m_<^4\rangle_{m_>}$

$C_3 = \langle m_>^1 \cdot m_<^2\, m_>^3 \cdot m_<^4\rangle_{m_>}$



$C_4 = \langle m_>^1 \cdot m_>^2\, m_>^3 \cdot m_>^4\rangle_{m_>}$

C_1 and C_4 are unimportant

indep of $m_>$ – just a number.

This is $\dfrac{\int Dm_>\, m_<^1\cdot m_<^2\, m_<^3\cdot m_<^4\, e^{-\beta H[m_>]}}{\int Dm_>\, e^{-\beta H[m_>]}}$

$= U[m_<]$ So C_2, C_3 are the only interesting ones.....

Now: $\langle m_i(q) m_j(q')\rangle_0$

$$= \frac{\int D\underline{m}(q)\, m_i(q) m_j(q')\, e^{-\beta H_0[\underline{m}(q)]}}{\int D\underline{m}(q)\, e^{-\beta H_0[\underline{m}(q)]}}$$

$$= \delta_{ij}(2\pi)^d\delta^d(q+q')\frac{1}{t+Kq^2} \quad G_0(q)$$

Lemma!

then: $C_2 = \underline{m}_<(q_3)\cdot\underline{m}_<(q_4)\; n\,(2\pi)^d\delta^d(q_1+q_2)\, G_0(q_1)$

(have taken these outside the > integral.)

$C_3 = \underline{m}_<(q_2)\cdot\underline{m}_<(q_4)(2\pi)^d\delta^d(q_1+q_3)\, G_0(q_1)$

So: the new Hamiltonian after coarse graining is same as old one apart from a change in the quadratic coeff. t:

$$t \to \tilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}G_0(q)\;!$$

② Rescale: $q' = bq$ ③ Renormalise $m' = \dfrac{m_<(q')}{z}$

then:

$$\beta H'[m'(q')] = \int_0^\Lambda (dq')\, b^{-d} z^2\left(\frac{\tilde{t}+Kb^{-2}q'^2}{2}\right)|m'(q')|^2$$

$$+ uz^4 b^{-3d}\int_0^\Lambda (dq_1')(dq_2')(dq_3')\, \underline{m}'(q_1')\cdot\underline{m}'(q_2')\,\underline{m}'(q_3')\cdot\underline{m}'(\Sigma - q_i')$$

then:

$t' = b^{-d}z^2\tilde{t}$ $K' = b^{-d-2}z^2K$ $u' = b^{-3d}z^4u$

so if $K' = K$, z must $= b^{1+d/2}$ and we get the $t = \; = 0$ fixed point as before

$\Rightarrow$ $t' = b^2\tilde{t}$ and $u' = b^{4-d}u$

these are discrete recursion relations – let us set

$b = e^{l} \simeq 1 + \delta l$:

$t' = t(b) = t(1+\delta l)$

$= t\big| + \delta l \dfrac{dt(l)}{dl}\Big|_{l=1}$

and

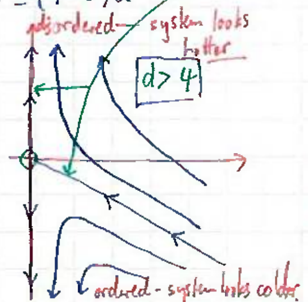
$u' = u + \delta l \dfrac{du}{dl}$

surface area of a sphere

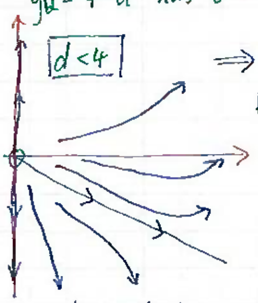
$$\text{So } \frac{dt}{dl} \simeq 2t + \frac{4u(n+2)S_d\Lambda^d}{(t+K\Lambda^2)(2\pi)^d}$$

and

$$\frac{du}{dl} \simeq (4-d)u$$



only one relevant direction ∴ describes phase transition



no! two relevant dir'ns

Oh no no no....

let $t \simeq t^* + \delta t$ and $u \simeq u^* + \delta u$ near a fixed point:

$$\frac{d}{dl}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix}2 & \dfrac{4(n+2)S_d\Lambda^{d-2}}{K(2\pi)^d}\\ 0 & 4-d\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}$$

So eigenvals are 2 and $4-d$! no diff to Gaussian

but eigen directions are different:

$y_t = 2$ is still assoc. with t dir'n ($u=0$)

$y_u = 4-d$ has $t = \dfrac{4u(n+2)K_d\Lambda^{d-2}}{(2-d)K}$

$\Longrightarrow$ so we have learnt little?

the series is alternating so maybe ∃ another F.P. at higher order.....

find new f.p. at $u = 4-d$ but u must be small ⚡ so let $\varepsilon = 4-d$!! so d is continuous??!!

Topological Phase Transitions

Consider n-component spins $\underline{S}_i = (s_1^i, s_2^i, \ldots s_n^i)$ $\underline{S}_i^2 = 1$

$$-\beta H = +K\sum_{\langle ij\rangle}\underline{S}_i\cdot\underline{S}_j = -\frac{K}{2}\sum_{\langle ij\rangle}\left((\underline{S}_i-\underline{S}_j)^2 - 2\right)$$

Ferromag

near $T=0$, $\underline{S} \simeq$ const and Fluct. are low energy $= (0,0,\ldots,1)$

so $-\beta H[S(x)] = -\beta E_0 - \dfrac{K}{2}\int d^dx\,(\nabla\underline{S})^2$

and this gives Z for the non-linear σ-model:

$$Z = \int D\underline{S}(x)\,\delta(S^2-1)\, e^{-\beta H[S]}$$

Parameterise the $n-1$ transverse Goldstone modes by $S(x) = (\underline{\pi}(x), \sqrt{1-\pi^2})$ and subst in $H[S]$

get to quadratic order:

$\langle\pi(x)\pi(-x)\rangle = a^{2-d} \propto$ Temp $\quad d>2$

$L^{2-d} \quad d<2$

can always find temp where fluct. are small.

→ Consider $n=2$, let $S = (\cos\theta, \sin\theta)$ then Hamiltonian is $-\beta H = K\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)$

For high temp, can expand in powers of K:

$$Z \simeq \int_0^{2\pi}\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_N}{2\pi}\prod_{\langle ij\rangle}\left(1+K\cos(\theta_i-\theta_j)+O(K^2)\right)$$

now: $\displaystyle\int_0^{2\pi} d\theta_1\cos(\theta_1-\theta_2) = 0$ and $\displaystyle\int_0^{2\pi} d\theta_2\cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3) = \frac{1}{2}\cos(\theta_1-\theta_3)$

↳ so terms in product that end give zero

So correlation f'n $\langle S(0)\cdot S(r)\rangle$

$= \langle\cos(\theta(0)-\theta(r))\rangle$

$\sim \left(\dfrac{K}{2}\right)^{|r|} = e^{-|r|/\xi}$

– only configurations joining 0 to r contribute

Low temp – Get Goldstone modes Hamiltonian:

$$-\beta H = \frac{K}{2}\int d^2x\,(\nabla\theta)^2$$

Now,

$$\langle S(0)\cdot S(r)\rangle = \mathrm{Re}\langle e^{i(\theta(0)-\theta(r))}\rangle$$

$$= \mathrm{Re}\left(\exp\left[-\tfrac{1}{2}\langle(\theta(0)-\theta(r))^2\rangle\right]\right)$$

$-\frac{1}{2}\ln\left(\frac{|r|}{a}\right)/\pi K$ $\quad = \left(\dfrac{a}{r}\right)^{\frac{1}{2\pi K}}$ ie Power law decay

Ph Tr ⑥

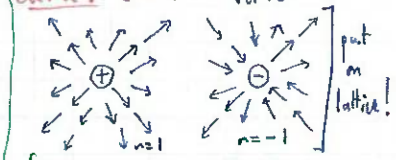
Topological Ph Tr Cont'd

X-Y model: High temp → exp; Low temp → power. ⇒ finite temp ph tr.... (self similarity....)

all of above allows ∃ ph tr.... or 2 comp spin! but none of it restricted to 2 dimensions!

Can study the relevance of interactions between Goldstone modes ie $(\nabla\theta)^4$ within RG: get $T=0$ fixed point only stable for $n=2$ in $d=2$. So in $d=2$, $n=2$ there is quasi-long range order for $t<0$

Gradient expansions apply to configs close to ground state – can be continuously deformed into) But topological defects cannot: vortex:



$\oint \nabla\theta \cdot dl = 2\pi n$ far from centre – continuum approx tangential, same all way round ∴ $\nabla\theta = \frac{n}{r} e_\theta$

Split energy into core $(r<a)$ and rest $(r>a)$:

$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a^L d^2x (\nabla\theta)^2 \rightarrow \pi K n^2 \ln\left(\frac{L}{a}\right)$$

dominant part – diverges with sys. size.... no defects form close to $T=0$....

For one vortex of charge n,

$$Z_1(n) \simeq \left(\frac{L}{a}\right)^2 \exp[\quad]$$

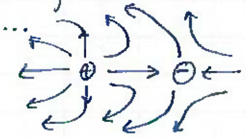
entropy factor: take up to make free energy.....

For $T\to 0$, K big, energy dominates, $Z\to 0$.

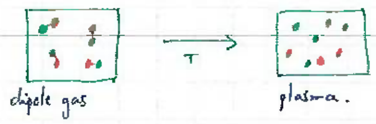
as $T\uparrow$, entropy may be s.t. defects can form... for $K > \frac{2}{\pi}$

Actually, defects form at larger K (smaller T) coz get dipoles – superpose $\nabla\theta_+ + \nabla\theta_- \simeq \frac{2d}{|r|^2}$ for sep. d....

energy of when $\int d^2x$ is finite ∴ appears at any temp....



Two phases:



Feynman Path Integral In Quantum Mechanics

S.E. $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$ integrate → $|\psi(t')\rangle = \underbrace{e^{-\frac{i}{\hbar}\hat{H}(t'-t)}}_{U}|\psi(t)\rangle$

Whack on left with $\langle x'|$ and insert identity to get it in pos'n rep:

$$U(x',t';x,t) = \langle x'|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|x\rangle, \quad \psi(x',t') = \int dx\, U(...)\,\psi(x,t)$$

let $\hat{U}(t',t) = \hat{U}(t',t_{N-1})\hat{U}(t_{N-1},t_{N-2})\cdots\cdots\hat{U}(t_1,t)$

discretise... then in the position rep: (insert lots of identities....)

$$U(x',t';x,t) = \int dx_{N-1}\cdots dx_1\, U(x',t';x_{N-1},t_{N-1})\cdots\cdots U(x_1,t_1;x,t)$$

Now $\langle x_{k+1}|\hat{U}(t_{k+1},t_k)|x_k\rangle = \langle x_{k+1}|\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|x_k\rangle$ as usual....

$$= \int\frac{dp_k}{2\pi\hbar}\langle x_{k+1}|p_k\rangle\langle p_k|\exp -\frac{i}{\hbar}\hat{H}\Delta t|x_k\rangle$$

$$\exp\left(\frac{i}{\hbar}\left(p_k\dot{x}_k - H(p_k,x_k)\right)\Delta t\right) + O(\Delta t^2)$$

normal ordering....!

So the full prop. is:

$$U(x',t';x,t) = \int\frac{dp_{N-1}}{2\pi\hbar}\cdots\frac{dp_0}{2\pi\hbar}\int dx_{N-1}\cdots dx_1 \exp\left[\frac{i}{\hbar}\sum_{k=0}^{N-1}\left(p_k\dot{x}_k - H(p_k,x_k)\right)\Delta t\right]$$

$$\therefore U = \int \mathcal{D}x(t'')\,\mathcal{D}p(t'')\,e^{\frac{i}{\hbar}S(p,x)} = \int \bar{\mathcal{D}}x(t'')\,e^{\frac{i}{\hbar}S(x)}$$

in limit $N\to\infty$ $\Delta t\to 0$

For Gaussian Ham. can do the p integral – get consts, put into $\mathcal{D}x$ then get: $e^{\frac{1}{2}m\dot{x}^2}$ from completing the square.

So
$$S(p,x) = \int_t^{t'} dt''\left(p\dot{x} - H(p,x)\right)$$
or
$$S(x) = \int_t^{t'} dt''\left(\frac{m\dot{x}^2}{2} - V(x)\right)$$

and $\bar{\mathcal{D}}x \equiv \lim_{\substack{N\to\infty \\ \Delta t\to 0}}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{N/2} dx_1\cdots dx_{N-1}$

for a free particle, $V=0$, and

$$U(x',t';x,t) = \left(\frac{m}{2\pi i\hbar(t'-t)}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x'-x)^2}{(t'-t)}}$$

Path Integral In Statistical Mechanics

Take $U(x',t';x,t)$ let $t' = -iT$, $t'' = -i\tau$, $t = 0$

note! plus sign

Then
$$U(x',T,x,0) = \int \mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar}\int_0^T d\tau\left[\frac{m}{2}\left(\frac{\partial x}{\partial\tau}\right)^2 + V(x(\tau))\right]}$$

[Note now the points t_k can lie in the complex plane!]

The plus sign means $[\quad] = E_{tot}$!!

and Classical statistical mechanics says:

$$Z = \int \mathcal{D}x(\tau)\, e^{-\beta E_{tot}}$$ so let $\beta = \frac{1}{\hbar}$

Path Integral In Quantum Stat. Mech.

We know $Z_{qu} = \mathrm{Tr}\, e^{-\beta H} = \int dx \langle x|e^{-\beta H}|x\rangle$

$$= \int dx\, U(x, t'=-i\beta\hbar; x, t=0)$$

ie let $t' \longrightarrow -\frac{i\hbar}{kT}$ and propagate from $x(0)$ back to the same position after $t' = -i\beta\hbar$ then integrate over all x to get Z_{qu}.

Ph Tr ⑦

$$G(x, x'; T) = \int \mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar} S[x(\tau)]}$$ $x(0) = x$, $x(T) = x'$

$$S[x(\tau)] = \int_0^T d\tau \left(\frac{m}{2} \dot{x}^2(\tau) + V[x(\tau)] \right)$$ ($\mathcal{L}^{euc}$)

Particle In a Single Potential Well

NB In the Euclidean action, can express $x(\tau)$ as $= \bar{x}(\tau) + \delta x(\tau)$ then using a complete set $\{x_n(\tau)\}$, let: $\delta x(\tau) = \sum_n c_n x_n(\tau)$

where $\int_0^T d\tau\, x_n(\tau) x_m(\tau) = \delta_{nm}$

and $x_n(0) = x_n(T) = 0$!

The Jacobian for $\mathcal{D}x(\tau) \to \mathcal{D}x_n(\tau)$ will usually be 1 but cancel anyway.....

subst $x \to \bar{x} + \delta x$, get:

$$S[\bar{x}(\tau)] + \int_0^T d\tau \left[\frac{\partial \mathcal{L}^{euc}}{\partial x}\Big|_{\bar{x}} \delta x + \frac{\partial \mathcal{L}^{euc}}{\partial \dot{x}}\Big|_{\dot{\bar{x}}} \delta \dot{x} \right]$$

zero because these are the E-L eq'ns! we are expanding about stat pt

$$+ \frac{1}{2} \int_0^T d\tau \left[\frac{\partial^2 \mathcal{L}^{euc}}{\partial x^2}\Big|_{\bar{x}} \delta x^2 + \frac{\partial^2 \mathcal{L}^{euc}}{\partial \dot{x}\, \partial x}\Big|_{\bar{x}} \delta x \delta \dot{x} + \frac{\partial^2 \mathcal{L}^{euc}}{\partial \dot{x}^2}\Big|_{\dot{\bar{x}}} \delta \dot{x} \right] + O(3)$$

$\frac{\partial^2 \mathcal{L}^{euc}}{\partial x^2} = V''(\bar{x})$; cross term $\to 0$ as $\mathcal{L}^{euc}$ has no cross terms; $\frac{\partial^2 \mathcal{L}^{euc}}{\partial \dot{x}^2} = m$; $O(3)$: not for Gaussians

$m\ddot{\bar{x}} = +V'(\bar{x})$ note the sign!! (E-L)

$$\therefore S[x(\tau)] = S[\bar{x}(\tau)] + \frac{1}{2} \int \left(m\, \delta\dot{x}^2 + V''(\bar{x})\, \delta x^2 \right) d\tau$$

$$= \bar{S}_{[x_0]} + \frac{1}{2} \int \delta x \underbrace{\left(-m\, \partial_\tau^2 + V''(\bar{x}) \right)}_{M^{-1}} \delta x$$

$$= S_{cl} + \frac{1}{2} \sum_n \lambda_n c_n^2$$ and $\mathcal{D}x(\tau) \to \prod_n dc_n \times J$

So G is a product of Gaussians - get:

$$G(x'; T; x, 0) = J e^{-\frac{S_{cl}}{\hbar}} \prod_n \frac{1}{\sqrt{\lambda_n}} \left(1 + O(\hbar) \right)$$

(J: some factor; $O(\hbar)$: cubic terms!)

Classical eq'n of motion for a particle in potential $-V(x)$

$\therefore$ Energy $E = \frac{1}{2} m \dot{x}^2 - V(x)$

So if $V = \cup$ then $-V = \cap$ and $\bar{x} = 0$ is the only solution satisfying the boundary conditions. Can evaluate $\prod_n \lambda_n$ - compare with free particle: $\omega = 0$

$$\prod_n \frac{\lambda_n(\omega = 0)}{\lambda_n(\omega = \omega)} = \prod_n \left(\frac{1}{1 + \frac{\omega^2 T^2}{n^2 \pi^2}} \right)^{1/2} = \sqrt{\frac{\omega T}{\sinh(\omega T)}}$$

$$\text{So, } G(0, 0; T) = \left(\frac{m\omega}{2\pi\hbar \sinh(\omega T)} \right)^{1/2} \left(1 + O(\hbar) \right)$$

Now can solve for $\bar{x}$ and subst back $\to S_{cl}[\bar{x}]$ to get:

$$S_{cl} = \frac{m\omega}{2} \left((x^2 + x'^2) \coth(\omega T) - \frac{2xx'}{\sinh(\omega T)} \right)$$

let $T \to \infty$ (low temp...) then get:

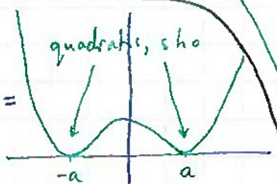
$$G(0, T; 0, 0) = \sqrt{\frac{m\omega}{\pi}}\, e^{-\frac{\omega T}{2}} e^{-\frac{m\omega^2}{2}(x^2 + x'^2)}$$

$$= \sum_m e^{-\frac{E_m T}{\hbar}} \psi_m^*(x) \psi_m(x') \text{ by def'n}$$

- have found E_0 and $\psi_0(x)$!!

The Double Well

Potential is: $V(x) = \frac{m\omega^2}{8a^2} (x^2 - a^2)^2 =$



Classical solutions are: particle stays at $-a$, a (NB pot is $-V$)

There is another solution: $G(a, -a; T) = \langle a | e^{-HT/\hbar} | -a \rangle$

So take the semi classical (ie Gaussian) approx again, as $T \to \infty$, $E \to 0$ so: $\dot{\bar{x}} = \sqrt{\frac{2V}{m}}$ $\longrightarrow$ can integrate to get $\tau(\bar{x})$

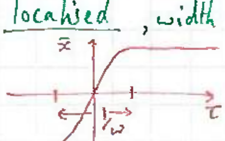
Solutions are instantons

Instanton action: $S_{inst} = \int d\tau\, m\dot{\bar{x}}^2 = \int_{-a}^{a} dx \sqrt{2mV(x)}$ from $\mathcal{L}^{euc}$

Close to $\bar{x} = a$ ie $\tau \to \infty$!! can expand as Taylor series:

$\dot{\bar{x}} = \omega(a - \bar{x}) + O((\bar{x} - a)^2)$ $\therefore$ use to see that $a - \bar{x} \propto e^{-\omega T}$ $\longrightarrow$ they are localised, width $\frac{1}{\omega}$

actually: $\bar{x} = a \tanh\left[\frac{\omega}{2} \left(\tau - \frac{T}{2} \right) \right]$



Local nature means that dilute gas of instanton/anti-i pairs is also a solution of the motion. - with action $S = n S_{inst}$.

Get:

$$G(a, \pm a; T) = \sqrt{\frac{m\omega}{\pi}}\, e^{-\frac{\omega T}{2}} \sum_{n = \text{even/odd}} \frac{\left(K e^{-\frac{S_{inst}}{\hbar}} T \right)^n}{n!} \left[1 + O(\hbar) \right]$$

if no instantons get correct result - evolve between kinks; K: some factor; $e^{-S_{inst}/\hbar}$: evolve through kink; $T^n/n!$: from integrate over all poss. pos'ns of kinks

giving:

$$G(a, \pm a; T) = \sqrt{\frac{m\omega}{\pi}}\, e^{-\frac{\omega T}{2}} \cdot \frac{1}{2} \underbrace{\left(e^{K e^{-\frac{S_{inst}}{\hbar}} T} \mp e^{-K e^{-\frac{S_{inst}}{\hbar}} T} \right)}_{\text{ie sinh or cosh....}} (1 + O(\hbar))$$

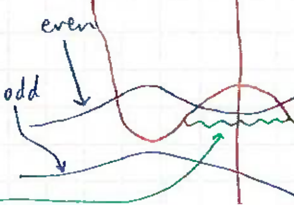
So energies of two lowest states are:

$$E_\pm = \frac{1}{2} \hbar\omega \pm \hbar K e^{-S_{inst}/\hbar}$$

$\longrightarrow$ very small but the leading order contribution to $E_+ - E_-$ ....

ie two even and odd combinations of harmonic oscillator single well solutions but degeneracy broken by tunneling!

[NB can use WKB but not for fields....]



$S = n S_{inst}$ etc only works if instantons are well separated - let's look at density of instantons: $\frac{n}{T}$: for $\sum_n \frac{y^2}{n!}$ (should be y^n?)

then dominant contribution is from $n \sim y$

ie $n \lesssim KT \exp(-S_{inst}/\hbar)$

$\therefore \frac{n}{T}$ is exponential small!

Ph Tr ⑧

1-d Ising Model with nearest neighbour interaction

$J_{ij} = \frac{\delta_{i,j\pm 1}}{T}$

Hamiltonian is: $-\beta H = \sum_{ij} J_{ij} S_i S_j$

ferromag $\to J_{ij} > 0$.

can express energy in terms of no. of domain walls, r:

$$Z = 2 \sum_r {}^{N-1}C_r \, e^{\frac{N-2r}{T}}$$

Stirling: $\frac{(N-1)!}{r!(N-1-r)!} \sim \frac{N^r}{r!} \sim N^r e^{\left[-r \ln\left(\frac{r}{e}\right)\right]}$

So $Z \simeq e^{N/T} \sum_r N^r e^{-\frac{2r}{T} - r\ln\left(\frac{r}{e}\right)}$

So in saddle pt (MFT) approx, vary free energy w.r.t. r get no of domain walls: $r_* = N e^{-2/T}$

if $N = \infty$, no LRO $\therefore$ no ph tr.

In finite system, $T_c \simeq \frac{2}{\ln N}$

1-d Ising Model with longer ranged interaction

isomorphic to double well.

$J_{ij} = \frac{e^{-K|i-j|}}{T}$

$\to$ to calculate the partition function, use a Hubbard Stratonovich transformation

write:

$$e^{\sum_{ij} S_i J_{ij} S_j} = C \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\phi_k \, e^{-\sum_{ij} \phi_i J^{-1}_{ij} \phi_j - 2\sum_i S_i \phi_i}$$

Must invert J_{ij}:

Go to Fourier rep:

$$\sum_{ij} S_i J_{ij} S_j = \int \frac{dq_1}{2\pi}\frac{dq_2}{2\pi} S(q_1) S(q_2) \sum_{ij} \frac{e^{-K|n_i - n_j|}}{T} e^{iq_2 n_j - iq_1 n_i}$$

let $n = n_i - n_j$, $\bar{n} = n_j$, then get

$$\sum_{n,\bar{n}} \frac{e^{-K|n|}}{T} e^{i(q_2+q_1)n_j - iq_1(n_i - n_j)}$$

$\to \delta(q_1 + q_2)$

$$= \int \frac{dq}{2\pi} |S(q)|^2 J(q) \quad \text{where:}$$

$$J(q) = \sum_{n=-\infty}^{\infty} \frac{e^{-K|n| - iqn}}{T}$$

can easily do the sum to get

$$J(q) = \frac{1}{T\left(\underbrace{\coth(K)}_{c} - \underbrace{\frac{1}{\sinh(K)}}_{b}\cos(q)\right)}$$

$\to$ now $\sum_{ij} \phi_i J^{-1}_{ij} \phi_j = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{|\phi(q)|^2}{J(q)}$

so

$$J^{-1}_{ij} = T \begin{pmatrix} c & -b/2 & 0 & 0 & \cdots \\ -b/2 & c & -b/2 & 0 & \cdots \\ 0 & -b/2 & c & -b/2 & \cdots \\ 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

shorter range interaction gives more complicated matrix......

which gives for the partition function:

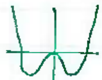
$$Z = C \int \prod_{k=1}^{N} d\phi_k \exp\left[\underbrace{-\frac{bT}{2}\sum_i (\phi_{i+1} - \phi_i)^2}_{\text{Kinetic term}} \underbrace{- \sum_i T(c-b)\phi_i^2 - \ln(2\cosh(2\phi_i))}_{\text{potential term } U(\phi_i)}\right]$$

Mass (i.e. t) $= \left.\frac{d^2 U}{d\phi^2}\right|_{\phi=0}$ for $t > 0$ and $U =$  , $t = 2[(c-b)T_c - 2]$

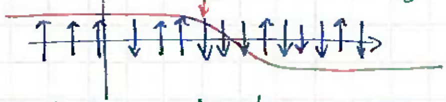
high temp. phase $T \gg \frac{1}{K}$

and $c - b = \tanh\left(\frac{K}{2}\right)$

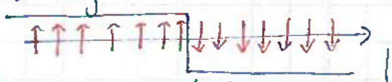
Might think:

Low temp phase, symm. is broken: $U =$  but symm is restored ie LRO destroyed.

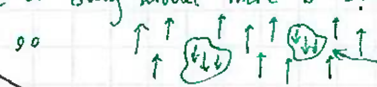
as instanton configs connect degenerate vacua:



[For nearest neighbour:



In 2-d Ising model there is LRO as the domains are small (short range int.)



A.Q.F.T. ①

S-Matrix

→ Adiabatically switch off ← NO CAN DO when have zero mass particles → I.R. divergences !!
coupling : (time)

hope: free fields satisfy

$[\phi^{in}(\underline{x},t), \dot\phi^{out}(\underline{y},t)] = i\delta^{(3)}(\underline{x}-\underline{y})$ ← ETCCR

Actually get $\frac{i}{Z}\delta^{(3)}(\underline{x}-\underline{y})$ ← renorm. const

all others zero

Solution of: $(\partial^2+m^2)\phi^{in/out}(x) = 0$

$\phi^{in/out}(x)$ defined for all time !!

Can make create/anih. ops:] time indep

$$\phi_\alpha^{in/out} = \frac{1}{i}\int d^3x\, \phi^{in}(x)\overleftrightarrow{\partial^0} f^*_\alpha(x)$$

Can expand: $\phi^{in/out}(x) = \sum_\alpha \left(f_\alpha(x)\phi_\alpha^{in/o} + f_\alpha^*\phi_\alpha^{i/o\dagger}\right)$ by orthogonality.

Subst this in ETCCR and get: $[\phi_\alpha^{in/o}, \phi_\beta^{in/o\dagger}] = \delta_{\alpha\beta}$

This is a wavepacket generalisation of $a_k, a_k^\dagger$ creating plane wave states whose wavef'ns are $e^{\pm ik\cdot x}$. Here we have wavpacket wave-f'ns, $f_\alpha(x)$

Introduce complete set of solutions of: $\{f_\alpha(x)\}$
with +ve energy: $(\partial^2+m^2)f_\alpha(x) = 0$ / $(-k^2+m^2)f_\alpha(k) = 0$

So $f_\alpha(x) = \int (dk)\, 2\pi\delta^+(k^2-m^2) f_\alpha(k) e^{-ik\cdot x}$ ← wavepackets

and $f_\alpha^*(x)$ is a -ve energy solution.

Orthonormality condition is

$$\int d^3\underline{x}\, f_\alpha(x)\overleftrightarrow{\partial^0} f_\beta^*(x) = i\delta_{\alpha\beta}$$

covariant - see by going to k space

Independent of time !!

Now STATES: assume ∃ vac such that obs $|0\rangle = 0$, $\langle 0|0\rangle = 1$

$|\alpha\, in\rangle = \phi_{\alpha_1}^{in\dagger}\phi_{\alpha_2}^{in\dagger}\dots|0\rangle$, $|\alpha\, out\rangle = \phi_{\alpha_1}^{out\dagger}\dots|0\rangle$

then:

$$|\alpha\, in\rangle = S|\alpha\, out\rangle \qquad SS^\dagger = \mathbb{1}$$

States are complete/orth. : $\langle\beta\, in|\alpha\, in\rangle = \sum_{perms}(\Pi\,\delta_{\alpha\beta})$
- defines S completely.

Single Particle States

no change: $|\alpha\, in\rangle = |\alpha\, out\rangle$ so drop labels !!

Know $\langle 0|\phi(x)|\alpha\, in\rangle$ satisfies K-G eq'n so can expand as:

$$= \sum_\beta c_\beta f_\beta(x) + d_\beta f_\beta^*(x)$$

From 4-mom operator P_μ:

$[P_\mu, \phi(x)] = -i\partial_\mu\phi(x) \iff e^{iP\cdot a}\phi(x)e^{-iP\cdot a} = \phi(x+a)$

Now use → use $\partial^2\phi = [P_\mu,[P^\mu,\phi]]$ and $P^2|\alpha\, in\rangle = m^2|\alpha\, in\rangle$ to see:

$$\langle 0|\phi(x)|\alpha\, in\rangle = f_\alpha(x)$$

let $x^0\to+\infty$, $\underbrace{\langle 0|\phi^{out}|\alpha\, in\rangle}_{f_\alpha} = \sum_\beta \underbrace{\langle 0|\phi^{out}|\beta\, out\rangle}_{f_\beta}\underbrace{\langle\beta\, out|\alpha\, in\rangle}_{\delta_{\alpha\beta}}$

So $|\alpha\, in\rangle = \sum_\beta |\beta\, out\rangle\delta_{\alpha\beta}$, QED!

Multiparticle States :

LSZ Reduction formula

eg 2 particle → 2 particle scattering:

want $\langle\alpha'\beta'\, out|\alpha\beta\, in\rangle$ $(= \langle\alpha'\beta'_{out}|S|\alpha\beta\, out\rangle)$

$= \langle\alpha'\beta'_o|\phi_\alpha^{in\dagger}|\beta\rangle \underset{Herm.}{=} -\frac{1}{i}\int d^3x\,\langle\alpha'\beta'\, out|\phi^{in}(x)|\beta\rangle\overleftrightarrow{\partial^0}f_\alpha(x)$ ∀ x^0

for $x^0 = -\infty$, $\phi^{in}(x) = \phi(x)$ and use cunning trick:

$$= i\underbrace{\int d^3\underline{x}\,\langle\alpha'\beta'\, out|\phi^{(out)}(x)|\beta\rangle\overleftrightarrow{\partial^0}f_\alpha(x)}_{x^0=+\infty} - i\underbrace{\int d^4x\,\partial^0\left[\langle\alpha'\beta'\, out|\phi(x)|\beta\rangle\overleftrightarrow{\partial^0}f_\alpha(x)\right]}_{x^0=-\infty}$$

just $\langle\alpha'\beta'\, out|\alpha\beta\, out\rangle = \sum_{perms}(\Pi\,\delta_{\alpha'\beta'}\delta_{\alpha\beta}\dots)$

Disconnected parts !

connected part....

$$= -i\int d^4x\, f_\alpha(x)(\partial^2+m^2)\langle\alpha'\beta'\, out|\phi|\beta\rangle$$

(using $\int d^3x\,\langle\phi\rangle\vec\nabla^2 f = +\int d^3x\,\langle\phi\rangle\vec\nabla^2 f$ (sphere at ∞) as f is w-packet)

now $\langle\alpha'\beta'|\phi(x)|\beta\rangle = \frac{1}{i}\int d^3y\, f_{\alpha'}^*\overleftrightarrow{\partial_y^0}\langle\beta'|\phi^{out}(y)\phi(x)|\beta\rangle$ ∀ y^0

So let $y^0 = +\infty$ and put in time order: $T\phi(x)\phi(y) = \phi(y)\phi(x)$ then:

transform to 4 mom-integral - adds to disconnected terms, do again for β', β etc then write $S = 1 + iT$, then T is connected parts: go to plane wave limit ie let $f_\alpha \to e^{ip_\alpha\cdot x}$

then: $\langle\beta'_o|T|\beta_i\rangle = i\int d^4x\, d^4y\, e^{-ip_\alpha\cdot x + ip_{\alpha'}\cdot y}(\partial_x^2+m^2)(\partial_y^2+m^2)$
$\times\langle\beta'|T\phi(x)\phi(y)|\beta\rangle$

So: ? in?

$$\langle\alpha'\beta'\, in|S|\alpha\beta\, in\rangle = \text{discon. terms} + \lim_{\text{on shell}}(p_\alpha^2-m^2)\cdots(p_\beta^2-m^2)\,\text{F.T.}\left(\langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle\right)$$

Particles (= external legs) = poles !

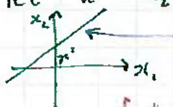
this must contain poles to cancel otherwise get 0 !

Mom. Cons. in Correlation f'ns

eg 2 pt f'n: $G(p_1,p_2) = i\int d^4x_1\, d^4x_2\, e^{ip_1\cdot x_1 + ip_2\cdot x_2}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$

apply 4-mom operator: → $\langle 0|T\phi(0)\phi(x_2-x_1)|0\rangle$

let $x' = x_2 - x_1$: $\int d^4x_1\, d^4x_2 = \int d^4x_1$ at const x' then $\int d^4x'$
the line ← scrolls line up and down !



Then get

$$\underbrace{i\int d^4x_1\, e^{ip_1\cdot x_1 + p_2\cdot x_1}}_{(2\pi)^4\delta^{(4)}(p_1+p_2)}\underbrace{i\int d^4x'\, e^{ip_2\cdot x'}\langle 0|T\phi(0)\phi(x')|0\rangle}_{G(p_2)}$$

For the n-pt function:

$$G(p_1\cdots p_n) = \delta^{(4)}(p_1+\cdots p_n)(2\pi)^4\, G(p_2\cdots p_n)$$

Re and Im parts of Correlation f'ns

For the 2 pt f'n can always remove one pos'n dep:

$$\tilde G(q_1) = i\int d^4x\, e^{-iq\cdot x}\langle 0|T\phi(0)\phi(x)|0\rangle = \underline{R + i\rho}\ \text{(expand T-prod!)}$$

Can write:

$R(q) = \frac{1}{2}i\int d^4x\, e^{-iq\cdot x}\,\varepsilon(x^0)\langle 0|[\phi(x),\phi(0)]|0\rangle$ where $\varepsilon = 1$ $x^0>0$, $=-1$ $x^0<0$

$\rho(q) = \frac{1}{2}\int d^4x\, e^{-iq\cdot x}\langle 0|[\phi(x),\phi(0)]_+|0\rangle$

Both R and ρ are real - can show (for Hermitian φ)

A.Q.F.T. ②

Lehmann Spectral Representation

To find contributions to 2 pt f'n from diff states, must insert the identity into eg $\rho(q)$:

Lorentz Invariance of GF's ie R and ρ:

- d^4x is Lor. Inv
- $e^{iq\cdot x}$ is Lor. Inv
- $\langle \phi_x \phi_0 \rangle$ is (1st term)
- Proper Lor. trans only changes sign of x^0 if $x^2 < 0$ for which commutator $= 0$!!! (by caus.)

$$\rho(q^2) = \frac{1}{2}\int d^4x\, e^{-iq\cdot x} \sum_{\alpha\, in} \left\{ \langle 0|\phi_x|\alpha\, in\rangle\langle \alpha\, in|\phi_0|0\rangle + \langle 0|\phi_0|\alpha\, in\rangle\langle\alpha\, in|\phi_x|0\rangle \right\}$$

for only p_α^0 +ve!

now $\phi_x = e^{iP\cdot x}\phi_0 e^{-iP\cdot x}$ and $e^{-iP\cdot x}|\alpha\, in\rangle = e^{-ip_\alpha\cdot x}|\alpha\, in\rangle$ then bring $\int dx$ through....

So:

$$\rho(q^2) = \frac{1}{2}\sum_\alpha |\langle 0|\phi_0|\alpha\, in\rangle|^2 \left\{\delta^{(4)}(q+p_\alpha) + \delta^4(q-p_\alpha)\right\}(2\pi)^4 \geq 0$$

(+ve energy; contributes for $-$ve q^0; $+$ve q^0)

So: Contributions to G.F. are: ① none from $|\alpha\, in\rangle$ = vacuum
② one particle state when $q^2 = (\text{one p. momentum})^2 = m^2$
③ Continuous dist'n of q for 2 or more particle states, $q^2 \geq (2m)^2$

Lehmann Spectral function $\equiv \mathrm{Im}\,(G(q))$

$$\tilde{G}(q^2) = i\int d^4x \langle 0|T\phi(x)\phi(0)|0\rangle e^{-iq\cdot x}$$ but can write Heaviside f'n as: $\theta(\tau) = \int_{-\infty}^{\infty} \frac{i}{2\pi} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}$

So

$$= i\left(\frac{i}{2\pi}\right)\int_{-\infty}^{\infty} d\omega \int d^4x \left[\langle\phi(x)\phi(0)\rangle \frac{e^{-i(q^0+\omega)x^0}}{\omega + i\epsilon} + \langle \phi(0)\phi(x)\rangle \frac{e^{-i(q^0-\omega)x^0}}{\omega+i\epsilon}\right]$$

($\theta(x^0)$; $\theta(-x^0)$; $\times e^{-iq\cdot x}$)

translate to $\langle\phi(0)\phi(-x)\rangle$

then send $x^\mu \to -x^\mu$ in first term, $\omega \to -\omega$ in both:

$$= \frac{-1}{2\pi}\int d\omega \int d^4x \langle \phi(0)\phi(x)\rangle \left[\frac{e^{i(q^0-\omega)x^0 - iq\cdot x}}{-\omega + i\epsilon} + \frac{e^{i(q^0+\omega)x^0 - iq\cdot x}}{-\omega + i\epsilon}\right]$$

let $\Omega_1 = \omega - q^0$ $\quad\Omega_2 = \omega + q^0$

$$= \frac{-1}{2\pi}\int d\Omega\int d^4x \langle\phi(0)\phi(x)\rangle\left[\frac{e^{-i(\Omega, q)\cdot x}}{-q^0 - \Omega + i\epsilon} + \frac{e^{i(\Omega, q)\cdot x}}{q^0 - \Omega + i\epsilon}\right]$$

+ve energy part of ρ, $\theta(\Omega)\rho(\Omega, q)$ sim'ly

So:

$$G(q^2) = \frac{-1}{\pi}\int d\Omega\, \theta(\Omega)\,\rho(\Omega^2 - |q|^2)\left[\frac{1}{-q^0 - \Omega + i\epsilon} + \frac{1}{q^0 - \Omega + i\epsilon}\right]$$

$$\frac{-2\Omega + 2i\epsilon}{-(q^0)^2 + \Omega^2 - 2i\epsilon\Omega}$$ ($2i\epsilon \to$ small; $2i\epsilon\Omega \to i\epsilon'$)

let $\sigma = \Omega^2 - |q|^2$

then:

$$G(q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma\, \rho(\sigma)}{\sigma - q^2 - i\epsilon'}$$ ← DISPERSION RELATION

Can do same for commutator !!!

Renormalisation in Perturbation Theory

Renormalisability: depends on superficial degree of divergence ie power of mom in numerator − that in denominator

- Super-renorm'able if: only finite no. of diags sup. div ← coupling const has +ve mass dimension
- Renorm'able if: finite no. of amplitudes sup. div but at any / all orders of P.T. ← " " has zero " " (eg e)!
- Non-renorm'able if: any amplitude diverges at high enough order ← " " has −ve " "

eg ϕ^4 theory: P+S. method

let $\phi_0 \to Z_\phi^{1/2}\phi$, $m_0 \to Z_m^{1/2} m$, $\lambda_0 \to Z_\lambda\lambda$

$$\mathcal{L} = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right) - \frac{\lambda\phi^4}{4!} - \frac{1}{2}\left[\underbrace{(1-Z_\phi)}_{\delta_\phi}(\partial\phi)^2 - \underbrace{(1-Z_m Z_\phi)}_{\delta_m} m^2\phi^2\right] - \underbrace{(Z_\lambda Z_\phi^2 - 1)}_{\delta_\lambda}\frac{\lambda\phi^4}{4!}$$

($\mathcal{L}_0$; $\mathcal{L}_{int}$; counter terms → new vertices...)

now switch off $\mathcal{L}_{int}$ adiabatically!

Now to fix Z_ϕ, Z_m and Z_λ with renormalisation conditions:

After self-energy insertions, the propagator is:

$$iD_F^{-1}(p^2) = p^2 - m^2 + \delta_m m^2 - \delta_\phi p^2 + \Pi(p^2)$$

$\Pi(p^2) = \Pi(m^2) + \Pi'(m^2)(p^2 - m^2) + (p^2 - m^2)\tilde{\Pi}_0(p^2)$

Renorm. Condition: D_F has pole at m with residue one:

want these terms to cancel

- will happen if: $\delta_\phi = \Pi'(m^2)$ and $\delta_m m^2 = \Pi'(m^2)m^2 - \Pi(m^2)$

leaving: $iD_F^{-1}(p^2) = p^2 - m^2(1 + \tilde{\Pi}_0(p^2))$ ($\tilde{\Pi}_0 \to 0$ as $p^2 \to m^2$)

For the vertex we have: $\Lambda(p_1 \ldots p_4) = \lambda + \lambda\delta_\lambda + \underbrace{\text{all vertex insertions}}_{\lambda V_x(p_1\cdots p_4)}$

Renorm condition: $\Lambda(s = t = u = \frac{4m^2}{3}) = \lambda$

ie $\lambda = \lambda(1 + \delta_\lambda + V_x)$ → can calculate δ_λ to a given order in P.T.!

But there is a problem! What coupling was used in the self energy / vertex insertions?? Everything is probably fine but there is a way that makes it more explicit:

eg ϕ^3 theory (in 6-d) Landshoff method:

Free propagator: $D_F(p^2) = \frac{i}{p^2 - m_0^2}$

Dress with self energy insertions

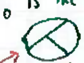
using bare coupling constant! so OK!!!

- leads to: $D_F'(p^2) = \frac{i}{p^2 - m_0^2 - \Pi(p^2)}$

expand Π to get $m_0^2 + \Pi(m^2) = m^2$

Then: $D_F'(p^2) = \frac{i}{p^2 - m^2}\cdot\frac{Z_\phi}{1 - Z_\phi\tilde{\Pi}_0(p^2)}$ ($\tilde{\Pi}_0 = 0$ at m^2)

where $Z_\phi = \frac{1}{1 - \Pi'(m^2)}$

now: $\tilde{\Pi}_0$ is the sum of diags like  each line is the propagator we derived. Can put Z_ϕ in numerator into the coupling by letting $g_0 \to g_1 = Z_\phi^{3/2} g_0$

- fine for internal vertices, but for the two external ones, each have Z_ϕ

So redefine $\tilde{\Pi}_0$: let $\tilde{\pi}_0 = \frac{\Pi_0}{Z_\phi}$

Now if let $\phi_0 = Z^{1/2}\phi$, then new prop. $D_F'(p^2)^{new} = \frac{i}{p^2 - m^2}\frac{1}{1 - \tilde{\Pi}_0}$

of $\langle 0|T\phi\phi|0\rangle$ as before!!! Now vertices:

for momenta $p_1\, p_2\, p_3$, $g =$  all insertions using D_F' new. ($\hookrightarrow = \Gamma_1$)

ie $g = g(p_1, p_2, p_3)$.

To fix g, need a renormalisation condition: let g be the value of the vertex insertions at $p_1^2 = p_2^2 = p_3^2 = m^2$.

and can write:

$$g = \Gamma_1(p_1, p_2, p_3) = g_1 + g_1\Gamma_2(m^2) + g_1\Gamma_3(p_1 p_2 p_3)$$

ie ... $= \lambda + $ ... ie expand Γ_1 about

impose condition: get $g = g_1\underbrace{(1 + \Gamma_2(m^2))}_{Z_1}$, then $g = \frac{Z_\phi^{3/2}}{Z_1} g_0$

- renormaliseable ↔ finite no. of counter terms!!

A.Q.F.T. (3)

also in ϕ^3 and that's it! no more div diags.

Dimensional Regularisation

eg $= \frac{1}{2} i g_0^2 \int \frac{d^6 k}{(2\pi)^6} \frac{1}{k^2 - m_0^2 + i\epsilon} \frac{1}{(p+k)^2 - m_0^2 + i\epsilon}$

Put denom. in exponent using Feynman params.

eg $\frac{1}{k^2 - m_0^2 + i\epsilon} = \frac{1}{i}\int_0^\infty d\alpha\, e^{i\alpha(k^2 - m_0^2 + i\epsilon)}$ — $i\epsilon$ ensures convergence!

Then complete the square in the exponent. If in Euclidean space, can sphericalise and do Gaussian Integral ... so WICK ROTATE over k!

ie consider k^0 integral: only contribution from poles: $Re(k^0)$

so can deform contour at will - rotate it 90° let $k^0 \to ik^0$, then are in Euc. space. k is large if any compt. is ...

we are left with $\int_0^\infty d\alpha\, d\beta$ - make $\int_0$ - have eg the subst:

$$\int_0^\infty d\alpha_1 \cdots d\alpha_r = \int_0^\infty \rho^{r-1} d\rho \int_0^1 dx_1 \cdots dx_r$$

where $x_1 + \cdots + x_r = 1$

- then extract a Gamma Function:

$$\Gamma(m) = \int_0^\infty dz\, z^{m-1} e^{-z}$$

Γ has poles at $0, -1, -2, \ldots$!

- can expand about 0 using Euler const or use property: $m\Gamma(m) = \Gamma(m+1)$

eg $\Gamma(2 - \frac{n}{2}) \to \Gamma(-1)$ $n \to 6$

$$\simeq \frac{2}{n-6}$$

$$= \frac{-g_0^2}{2(4\pi)^{n/2}} \Gamma\left(2 - \frac{n}{2}\right) \int_0^1 dx \left[m_0^2 - p^2 x(1-x)\right]^{\frac{n}{2}-2}$$

now - can extract divergent and convergent bits of $\Pi(p^2)$ by expanding [ ] in powers of $n-6$:

$$[\ ]^{\frac{n}{2}-2} = [\ ][\ ]^{\frac{n}{2}-3}$$

$$= [\ ] \exp\left[\left(\frac{n}{2} - 3\right)\log[\ ]\right]$$

can do $\int [\ ]$ easy

$$= [\ ]\left(1 + \left(\frac{n}{2} - 3\right)\log[\ ] + \left(\frac{n}{2} - 3\right)^2 \log^2[\ ]\right) + \ldots$$

then: $\propto \frac{1}{n-6}\left(m_0^2 - \frac{p^2}{6}\right)$ $\quad \propto \int [m_0^2 - p^2 x(1-x)] \log[\ ]$

now $\Pi = \bigcirc + \ominus + \ldots$

$$= \infty (g_0^2 + g_0^4 + \ldots)$$

So to lowest order in g_0, $\bigcirc = \Pi(p^2)$.

So can calculate eg $\Pi(m^2) \simeq \frac{-g_0^2}{(4\pi)^3} \frac{1}{n-6}\left(m_0^2 - \frac{m^2}{6}\right)$

$$\Pi'(m^2) = \frac{\partial}{\partial p^2}\left(\Pi(p^2)\right)\Big|_m \simeq \frac{g_0^2}{4\pi^3} \frac{1}{n-6}\left(\frac{-1}{6}\right)$$

and convergent part!, $(p^2 - m^2)\Pi_c(p^2) =$

Generating Functional

- divergent until say let $m^2 \to m^2 - i\epsilon$ then get factor $e^{-\frac{\epsilon}{2}} \to$ converges!

Free field case:

$$Z_f[J] = \int d\phi\, e^{i\int d^4x [\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi]} = \int d\phi\, e^{i\int d^4x [-\frac{1}{2}\phi \underbrace{(\partial^2 + m^2)}_{G^{-1}}\phi + J\phi]}$$

Evaluate explicitly: $Z_f[J] = \underbrace{(\sqrt{2\pi})^\infty (\det G)^{\frac{1}{2}}}_{Z_f[0]} e^{\frac{1}{2}i\int d^4x\, J(x)(\partial^2 + m^2 + i\epsilon)^{-1} J(x)}$ — already diag. $\to D_F$ - can see by let $J = FT\tilde{J}$

- let $\phi \to \phi + GJ$

So $Z_f[J] = Z_f[0] . e^{\frac{1}{2} i \int J D_F J\, d^4x}$

Now: $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^4(x-y)$ So
$$\boxed{D_F(x_1 - x_2) = \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \frac{Z_f[J]}{Z_f[0]}}$$
free propagator. $\langle 0 | T\phi_1\phi_2 | 0 \rangle$

Interacting field case:

$$Z[J] = \int d\phi\, e^{i\int d^4x [\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi + \mathcal{L}_{int}]}$$

$$= \int d\phi \left(1 + \int i\mathcal{L}_{int}\, d^4y + \int (i\mathcal{L}_{int})^2 dy\, dz + \ldots\right) \times e^{i\int d^4x [\mathcal{L}_{free}]}$$

ie $Z[0] = \langle e^{L_{int}} \rangle_0 Z_0[0]$

$$= \left(\Sigma \text{ vac. bubbles}\right) Z_{free}[0]$$

now:

$$\frac{\left(\frac{-i\delta}{\delta J(x_1)}\right)\left(\frac{-i\delta}{\delta J(x_2)}\right) Z[J]}{Z[0]}\Bigg|_{J=0} = \frac{\left(\Sigma \text{ all diagrams}\right) . Z_{free}[0]}{Z[0]}$$

but all diags = (connected diags) × (vac. bubbles coz combinatorics are OK.

So
$$\boxed{\langle 0 | T\phi(x_1)\phi(x_2) | 0 \rangle_c = \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \frac{Z[J]}{Z[0]}\Bigg|_{J=0}}$$

the "right" vacuum - keep only connected diagrams

- valid beyond P.T. - still need to renormalise

Generating Functional for Fermions

Fermionic green functions are antisymm if exchange external lines so no J

Instead use: anticommuting Grassmann Variables $[\sigma(x_1), \sigma(x_2)]_+ = 0$

also $\left[\frac{\delta}{\delta\sigma_1}, \frac{\delta}{\delta\sigma_2}\right]_+ = 0$ so $\frac{\delta}{\delta\sigma(x)} \sigma(y)\sigma(z) = \delta^4(x-y)\sigma(z) - \sigma(y)\delta^4(x-z)$

Free case:

$$Z_{free}[\sigma, \bar\sigma] = \int d\Psi\, d\bar\Psi\, e^{i\int d^4x [\bar\Psi(i\gamma\cdot\partial - m)\Psi + \bar\Psi\sigma + \bar\sigma\Psi]}$$

no $\frac{1}{2}$! coz not real

$$= Z[0,0]\, e^{-\int dx\, dy\, \bar\sigma(x) S_F(x-y) \sigma(y)}$$

and $\langle 0 | T\bar\Psi(x_1)\Psi(x_2) | 0 \rangle_c = \frac{-i\delta}{\delta\sigma(x_1)} \frac{\delta}{\delta\bar\sigma(x_2)} \frac{Z[\sigma, \bar\sigma]}{Z[0, \bar\sigma]}\Bigg|_{\bar\sigma = 0}$

Abelian Gauge Theory

$\mathcal{L}$ invariant under global $U(1)$

local - $\Rightarrow \exists A^\mu(x)$, gauge field

- ensures $\bar\Psi\gamma\cdot D\Psi$ is invariant.

Interaction term is $-e j_\mu A^\mu$ where $j_\mu = \bar\Psi\gamma_\mu\Psi$

e/m current is the local sym's Noether current!

Naively, expect the free gauge field $\mathcal{L}$ to be: $-\frac{1}{2}(\partial_\mu A^\nu)(\partial_\nu A^\mu)$

$$= -\frac{1}{2}\partial A^0 + \frac{1}{2}(\partial A)^2$$

sign difference means that in the Hamiltonian, the timelike bit cancels the longitudinal polarisation!

Better to have it in gauge invariant form:

$$\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- only differs from by divergence

so no prob when $\int$.

NB: $[D^\mu, D^\nu] = ieF^{\mu\nu}$

and $F^{0i} = -E^i$ and $F^{ij} = \epsilon^{ijk} B^k$

P.T.O. for $SU(2)$ / $SU(3)$ ....

AQFT ④

Non-Abelian Gauge Theory

$SU(2)$: ψ is an isodoublet $\binom{u}{d}$ (Y.M) ie vector in 2-D isospin space where u and d are 4-spinors. no!

So: $\bar\psi(i\gamma\cdot D - m)\psi = \bar u(\ )u + \bar d(\ )d$

→ ∃ global SU(2) symm: $\psi \to e^{\frac{i}{2}\tau\cdot\theta}\psi$ ← 3 component vector in "Lie Alg.-space!?" ; Pauli matrices

$[\frac{1}{2}\tau^i, \frac{1}{2}\tau^j] = i\epsilon^{ijk}\frac{1}{2}\tau^k$ and $tr(\frac{1}{2}\tau^i\frac{1}{2}\tau^j) = \frac{1}{2}\delta^{ij}$

Now - make local: $U = e^{-\frac{1}{2}ig\tau\cdot\omega}$ $\left(= e^{\tau\cdot\Omega}\right)$

Cov. deriv: $D_\mu = \partial_\mu + \frac{1}{2}ig\tau\cdot A_\mu$ where A transforms as:

$\tau\cdot B^\mu \to U\tau\cdot B^\mu U^{-1} - (\partial^\mu U)U^{-1}$ where $B^\mu = \frac{1}{2}igA^\mu$

for infinitesimal transformations, $A^\mu \to A^\mu + \partial^\mu\omega + g\omega\wedge A^\mu$

then $[D^\mu, D^\nu] = \frac{1}{2}ig\tau\cdot F^{\mu\nu} \Rightarrow F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - gA^\mu\wedge A^\nu$

Lagrangian: $\mathcal{L}_{YM} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}$ ← Gauge Covariant ie $\tau\cdot F \to U\tau\cdot F U^{-1}$

↑ Gauge Invariant

SU(3) - QCD.

SU(3) generators: $\lambda^a = 2t^a$

$D_\mu = \partial_\mu + \frac{1}{2}ig\lambda^a A^a_\mu$ ← Flavour labels

But the SU(3) tr'n acts on the colour labels (3 dim space r g b...) same QM except exclusion princ.

now $[t^a, t^b] = if^{abc}t^c$, $tr(t^at^b) = \frac{1}{2}\delta^{ab}$

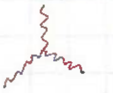
totally antisymm. ↑ not cross product now...

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$ not "free field" as F contains coupling!!

$\to \mathcal{L} = \sum_{\text{flavours } a} \bar\psi_a(i\not{D} - m_a)\psi_a - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}$

free field KE $+ gf^{abc}A^a_\mu A^b_\nu \partial^\mu A^{c\nu}$ (quad. terms)

$-\frac{1}{4}g f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}$



gluon self interaction!! (direct)

g ≡ colour charge.

Quantisation of Gauge Theories (Canonical)

Try canonical approach... let $A^\mu_{in}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} a^\mu(k)\, 2\pi\delta^+(k^2)$

But a^μ has 4 d.o.f. and we know gauge boson has spin 1 ie d.o.f. (massless) - so, let's see if can impose 2 constraints:

We know all observable quantities are gauge invariant - can remove 1 d.o.f. from A^μ by fixing the gauge:

eg 1: Axial gauge $n\cdot A(x) = 0$ — eg $n^\mu = (1,0,0,0)$ then $D^{00} = D^{0i} = D^{i0} = 0$ and $D_{ij} = \mathcal{O}[\delta^{ij} - \frac{k^ik^j}{k^2}]$

then prop. is: $\delta^{ab}\frac{i}{k^2+i\epsilon}\left[-g^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{n\cdot k} - \frac{n^2k^\mu k^\nu}{(n\cdot k)^2}\right]$

- breaks Lor. Inv. - have picked direction n in M^4

eg 2: Covariant Gauges: Landau Gauge: $\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right]$, $\partial_\mu A^\mu = 0$

and Feynman gauge $[-g^{\mu\nu}]$ ← (sq bracket only)

But: Probs: ETCCR's conflict with covariant gauge conditions - could let $\langle b|\,\partial_\mu A^\mu\,|c\rangle = 0$ instead though....

↳ this condition tells us which states are physical...

In axial gauges, one d.o.f. is removed eg $A^0 = 0$ but then lose eg Gauss's law: impose $\langle B|\,\text{div}\,E\,|C\rangle = 0$ again.....

But then still have 1 d.o.f. to exclude: ensure that prob physical state scattering to unphysical state is zero - this is basically O.K. for axial gauges

But for covariant gauges must introduce ghost field s.t.:

prob(final state is unphysical) + prob(ghost) = 0 - need indefinite metric

Fadeev - Popov Approach

Generating Functional:

$Z[J] = \int dA^{a\mu}\, e^{i\int d^4x\, \mathcal{L} + J^{a\mu}(x)A^a_\mu(x)}$ - must sort out two related difficulties:

① Assymptotically (ie set $g=0$), bit of $\mathcal{L}_{int}$ that remains is: $-\frac{1}{4}\int d^4x\,(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})^2$

$= -\frac{1}{4}\int d^4x\, A^a_\mu\,(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A^a_\nu$

inverse of... free field propagator - has no inverse!

ie F.T.: $(-g^{\mu\nu}k^2 + k^\mu k^\nu)k_\nu = 0$

② P.I. is very divergent as for each physical $A^{a\mu}$ ∃ ∞ others reached by gauge trn.

$\int dA$ is gauge invariant $\int dA\, f(A) = \int dA^\omega f(A)$

and can define $= \int dA\, f(A^\omega)$

a gauge inv measure over the group $U = e^{\frac{i}{2}g\tau\cdot\omega}$

$\int dU(\omega)\, f(U(\omega)) = \int dU(\omega)\, f(U(\omega)U(\omega_0))$

Now define:

$(\Delta[A,B])^{-1} = \int dU(\omega)\,\delta[F[A^\omega] - B]$

δ-final!

$\Delta[A,B] = \Delta[A^\omega, B]$ - gauge inv!

so can take in front of $U(\omega)$ integral at will:

$1 = \int dU(\omega)\,\delta[F(A^\omega) - B]\,\Delta[A^\omega, B]$

→ can fix the gauge in $Z[J]$ by inserting 1.... in $Z[J]$:

(writing $\Delta[A, F(A)] = \Delta[A]$) and $\int dB\,\Psi(B) = \int dU(\omega)\,\Delta[A^\omega]\,\Psi[F(A)]$

$Z[J] = \int\left(\int dB\,\Psi(B)\right)^{-1}\int \Delta[A^\omega]\, e^{iS[A] - \frac{i}{2\alpha}(F[A^\omega])^2}\, dA\, dU(\omega)$ where $\Psi(B) = e^{\frac{-i}{2\alpha}F(A^\omega)}$

(↑ includes $J\cdot A$)

now make a change of integration variable - all $A \to A^{-\omega}$

then $Z[J] = \int dU(\omega)\left(\int dB\,\Psi(B)\right)^{-1} dA\,\Delta[A]\, e^{iS[A] - J\cdot A^{-\omega} - \frac{i}{2\alpha}(F(A))^2}$

(↑ g inv.)

this is just const - cancels when normalise correlation functions.

Now, when $\frac{-i\delta}{\delta J(x_i)}$, pull down $\boxed{A^{-\omega}(x_i)}$! then to get S-matrix elements,

F.T. w.r.t. x_i, use LSZ: $\times q_i^2$ then take limit $q_i^2 \to 0$ (ie find residue)...

also take Lorentz scalar product with gauge boson polarisation vector $\epsilon_\mu(q_i)$....

now: Fspace: $A^{-\omega}_\mu = A_\mu(q_i) - iq_\mu\omega(q) - g\int\frac{d^4q'}{(2\pi)^4}\,\omega(q-q')\wedge A_\mu(q')$ ← convolution coz product in x-space.

transverse pol ∴ $\epsilon^\mu q_\mu = 0$

So S matrix element is:

$\langle s\rangle = \lim_{q^2\to 0}\left[q^2 I_\mu(q)\epsilon^\mu(q) - gq^2\int\frac{d^4q'}{(2\pi)^4}\,\omega(q-q')\wedge I_\mu(q')\epsilon^\mu(q)\right]$

↑ propagator has pole ∴ get non zero

here the pole is washed out by conv. → 0

→ so can just replace $A^{-\omega} \to A$ - works not just for infinitesimal.

A.Q.F.T. ⑤

Fadeev - Popov Cont'd

So if can let $A_\mu^{-\omega} \to A_\mu$ then can write:

$$Z[J] = \int dA\, \Delta[A]\, e^{iS[A] + J\cdot A - \frac{i}{2\alpha}F(A)^2} \cdot \int d\mu(\omega)$$

(now indep of ω; $\int d\mu(\omega)$ = volume of gauge gr.)

where:

$$(\Delta[A])^{-1} = \int d\mu(\omega)\, \delta[F(A^\omega) - F(A)]$$

→ in abelian theories, $A^\omega \simeq A + e\,\partial\omega$ for infinitesimal trans'ms. Then in Cov. gauges eg $F(A) = \partial_\mu A^\mu$ then $F(A^\omega) - F(A) = 0 \Rightarrow \partial^2\omega = 0$
$\Rightarrow \omega = 0$ everywhere as fields must vanish at ∞ + uniqueness thm.
For non abelian theories, $\exists$ other solutions corresponding to large gauge trans
– Gribov Copies.

We want P.T. so discard Gribov copies: Infinitesimally:

$$(\Delta[A])^{-1} = \int d\underline{\omega}\, \delta[(\partial_\mu \underline{\omega} + g\underline{\omega}\wedge \underline{A}_\mu)\frac{\delta F(A)}{\delta A_\mu}]$$ ($\underline{\omega}$?)

now just as $\delta(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}$,

$$\delta(\underline{f}) = \int d\underline{\eta}\, \exp\left(-i\int d^4x\, \underline{\eta}\cdot\underline{f} \times \frac{1}{(2\pi)^\infty}\right)$$

So

$$(\Delta[A])^{-1} = \text{Const.}\int d\underline{\omega}\, d\underline{\eta}\, e^{-i\int d^4x\, \underline{\eta}\cdot(\partial_\mu\underline{\omega} + g\underline{\omega}\wedge \underline{A}_\mu)\frac{\delta F(A)}{\delta A_\mu}}$$

Feynman Rules and Ghosts

From here take η and ω to be Grassman fields, then:

$$Z[J] = \int d\underline{A}\, d\underline{\eta}\, d\underline{\omega}\, e^{i\int d^4x(\mathcal{L}_{eff} + \underline{J}\cdot\underline{A})}$$

where: $\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \mathcal{L}_{GF} + \mathcal{L}_{FPG}$
(as before; gauge fix; F.P. ghosts)

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(F(A))^2 \quad \text{where } F(A) = \partial A \text{ or } n\cdot A$$

$$\mathcal{L}_{FPG} = -\underline{\eta}\cdot(\partial_\mu\underline{\omega} + g\underline{\omega}\wedge \underline{A}_\mu)\frac{\delta F}{\delta \omega}$$

↗ Cov: $-\underline{\eta}\cdot\partial^2\underline{\omega} - g\,\underline{\eta}\cdot\partial_\mu(\underline{\omega}\wedge \underline{A}^\mu)$
↘ Axial: $-\underline{\eta}\cdot n_\mu(\partial^\mu\underline{\omega} + g\underline{\omega}\wedge \underline{A}^\mu)$

→ η, ω behave as additional fields – not coupled to J so only have internal lines

Get Gauge Field Propagator from quadratic terms in $\mathcal{L}_{eff}$:

$$= -\frac{1}{2}A_\mu^a\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + \frac{1}{\alpha}\partial^\mu\partial^\nu\right)A_\nu^a$$

(this is the bit with zero e-val!; should cancel!)

$$\therefore D^{ab}_{\mu\nu}(k) = i\delta^{ab}\frac{\left(-g^{\mu\nu} + (1-\alpha)\frac{k^\mu k^\nu}{k^2}\right)}{k^2}$$

$\alpha = 1 \to$ Feynman Gauge so D simple.
$\alpha = 0$, Landau Gauge look at $\mathcal{L}_{GF} \Rightarrow \partial A = 0$
$\alpha = \infty$ not allowed as no Fix gauge! otherwise osc quick!

space!

Can see what $(\Delta[A])^{-1}$ is by going to discrete case:

$$= \int du_i\, dv_j\, e^{-iu_i M_{ij}^c v_j}$$ (product...; only symm. part contributes so diagonalise...)

$$= \int du_i' dv_j' e^{-i\sum_i \lambda_i u_i' v_i'} = \int dv_j\, 2\pi\delta(\textstyle\sum_j \lambda_j v_j)$$

$$= \frac{(2\pi)^N}{\prod_i \lambda_i} \propto \frac{1}{\det(\underline{M})}$$

Functional Determinants

But if u, v are Grassman Variables then get $\propto \det(M)$!! which is precisely what we want:

$f(u) = a + bu$, $\int du\, f(u) = b$! like diff...

eg $\int du_1 du_2\, f(u_1, u_2) = -c_{12}$ ...

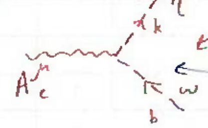
$$\text{So } \int du\, dv\, e^{-i\sum_i \lambda_i u_i v_i} = \int \prod du\,dv \sum_r \frac{(-i)^r}{r!}\left(\sum_i \lambda_i u_i v_i\right)^r$$ (prod!)

only term surviving is the one with all u_i, v_i in linearly:

$$\int du_1 \cdots du_n\, dv_1 \cdots dv_n (\lambda_1 u_1 v_1) \cdots (\lambda_n u_n v_n) = \text{const.}\det(\underline{M})$$

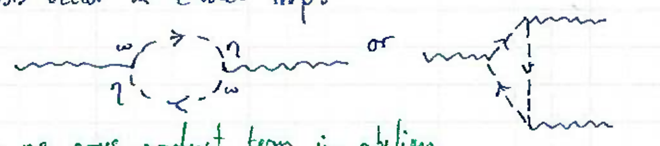
$$\mathcal{L}_{FPG} = -(\underline{\eta}\cdot\partial^2\underline{\omega} - g(\partial\underline{\eta})\cdot\underline{\omega}\wedge\underline{A}) \quad \text{cov gauge}$$

This changes $\omega \to \eta$ with propagator: $\frac{-i\delta^{ab}}{k^2}$



vertex: $g f^{abc} k^\mu$ — Ghosts

Ghosts occur in closed loops:



There is no cross product term in abelian gauge theory ∴ ghosts no interact with photons!
Just get $\int d\eta\, d\omega \exp(-i\int d^4x\, \eta\cdot\partial^2\omega)$ – just const factor!
QCD axial gauge: get $n_\mu\cdot(\ldots\ \underline{\omega}\wedge\underline{A}^\mu) = 0$
∴ ghosts disappear! Price is more complicated propagator

Renormalisation of Q.E.D.

Following Landshoff procedure, know vertices are ... so expect:

$e = Z_1 Z_2 Z_3^{1/2} e$ (Z_1 vertex insertions, Z_2 electron self energy, Z_3 photon self energy)

Electron propagator: $S_F(x) = (i\gamma\cdot\partial + m_0)\Delta_F(x)$

$$\hookrightarrow S_F(p) = i\frac{\not p + m_0}{p^2 - m_0^2 - i\epsilon}$$

but using $\not p^2 = p^2$ can write:

$$S_F(p) = \frac{i}{\not p - m_0 + i\epsilon}$$

then $S_F'(p) = \frac{i}{\not p - m_0 - \Sigma(p)}$ (old)

where $\Sigma(p) = $ –(1PI)–

let $\Sigma(p) = A + B(\not p - m) + \tilde\Sigma_0(p^2)$

NB different from scalar case!

$m = m_0 + A$

So $S_F'^{\,new}(p) = \frac{i}{\not p - m - \tilde\Sigma_0(p)}$ using e not e_0

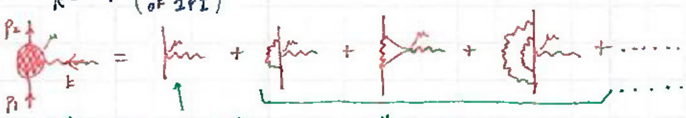
$Z_2 = \frac{1}{1-B}$ → Factor of Z_2 and put on e_0....

$\tilde\Sigma_0$ vanishes quadratically as go on shell.

Ward identity in the form k^μ (1PI) $= 0$ lets us extract tensor structure of $\Pi_{\mu\nu}(k)$: $= \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\Pi(k)$ (scalar) tensor structure collapses to give reg. sum:

total $\Pi^T_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2}\cdot\frac{Z_3}{1 - (\text{conv. bit of 1PI})}$ — take off Z_3 and put on e_0.....

Vertex Insertions:



$$-ie\Gamma^\mu(p_1, p_2) = -ie_0\gamma^\mu + -ie_0\Lambda^\mu(p_1, p_2)$$

now let $\Lambda^\mu(p_1, p_2) = \gamma^\mu L + (1 + L)\tilde\Lambda^\mu(p_1, p_2)$

then $-ie\Gamma^\mu(p_1, p_2) = -ie_0 Z_1(\gamma^\mu + \tilde\Lambda^\mu(p_1, p_2))$, $1 + L = Z_1$

then the on-shell renormalisation conditions mean that:

$-ie\Gamma^\mu(m^2, m^2) = -ie\gamma^\mu \Rightarrow e = Z_1 e_0$ and $\tilde\Lambda^\mu(p_1, p_2) = 0$ on shell

A.Q.F.T. ⑥

Ward Identity and Universality of electric charge

$S_F(p)\, S_F^{-1}(p) = \mathbb{1}$ — Dirac space.

$\therefore \left(\dfrac{\partial S_F(p)}{\partial p^\mu}\right) S_F^{-1}(p) = -S_F \dfrac{\partial S_F^{-1}}{\partial p^\mu}$

$= -S_F(-i\gamma^\mu)$

So:

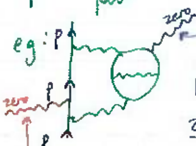
$$\boxed{\frac{\partial S_F(p)}{\partial p^\mu} = -S_F(p)(-i\gamma^\mu)S_F(p)}$$

i.e. $e_0 \dfrac{\partial}{\partial p^\mu}\left(\uparrow p \right) = -$ (zero mom photon inserted on fermion line p)

WARD IDENTITY.

look at -(1PI)- $= -i\Sigma(p)$

ie + + ...



eg: if add a zmph here get zero contrib as we $\int d(\text{loop mom})$

if add zmph here → $\dfrac{\partial \Sigma(p)}{\partial p_\mu}$

→ so: $e_0 \dfrac{\partial}{\partial p_\mu}\left(\text{1PI}\right) = -\left(\text{1PI with zero photon}\right) = -\Lambda^\mu(p,p)$

$= -\Lambda_c^\mu(p,p) + \tilde{\Lambda}_c^\mu(p,p)$ on shell

$= -ieL\gamma^\mu + (1+L)\tilde{\Lambda}_c^\mu(p,p)$ $x-1$

0 - renormalisation condition.

$\Sigma(p)$

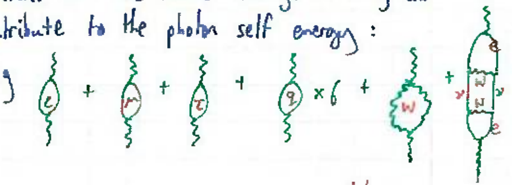
$= A + B(\gamma^\mu p_\mu - m_0) + \tilde{\Sigma}_0(p)$ — 0 on shell

so on shell, $e_0 B\gamma^\mu = -e_0 L\gamma^\mu$

ie. $\boxed{B = -L}$ $\rightarrow Z_1 = \dfrac{1}{Z_2}$

The fermion self energy and the fermion-photon vertex insertions cancel out! ie. only the photon self energy determines the measured electron charge → This is the connection between all the different particles with the same charge – they all contribute to the photon self energy:

eg (e) + (μ) + (τ) + (q) ×6 + (W) + ...



e-mag charge is the conserved charge associated with the conserved Noether current arising from the gauge inv. symmetry!!

Q.C.D. Renormalisation – general points

QED is "onshell" renormalisation so that if take renormalised theory to lowest order, get eg Rutherford formula but with m and e.
That's OK. as get lots of ~static QED situations.
But in QCD. ∃ confinement ∴ no can do......
Can get diff. schemes by including different finite bits in A, B, L (div) then $\Pi_c, \Sigma_c, \Lambda_c$ are diff. to compensate – all gives same if sum to all order but not if truncate!

Dimensional reg. preserves gauge invariance but ∃ freedom in def'n of γ matrices away from 4-d: can keep $[\gamma^\mu,\gamma^\nu]_+ = 2g^{\mu\nu}$ so that $p^2 = p^2$ still, but then $\text{tr}(\gamma^\mu\gamma^\nu) = g^{\mu\nu}\,\text{tr}\,\mathbb{1}$

for integer n.... even $= 2^{n/2}$, odd $2^{\frac{n-1}{2}}$

Anyway – can use any f'n of n provided that $f(4)=4$ (eg $f(n)=4$) corresponding to diff renorm. schemes.

For hard processes – energy/mom >> params → set quark mass = 0

then $i\gamma\cdot D + m_0 \rightarrow i\gamma\cdot D$ and

$\mathcal{L}^{QCD} = \bar\Psi i\gamma\cdot D\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{G.F.} + \mathcal{L}_{F.P.G.}$

for massless theories ∃ chiral symmetry + associated current – invariance under $\Psi \rightarrow \gamma^5\Psi$.

Dim-reg preserves chiral invariance, keeping quark mass = 0.

Q.C.D. Renormalisation – Minimal Subtraction

gives $g = Z_1 Z_2 Z_3^{1/2}\, g_0\, \mu^{\frac{n}{2}-2}$

- g: dimensionless coupling
- Z_1: vertex, Z_2: quark S.E. — cancel as before
- Z_3: gluon S.E.
- g_0: this has dimension $2-\frac{n}{2}$ mass
- $\mu^{\frac{n}{2}-2}$: this cancels g_0's dim's; arbitrary mass – the renormalisation scale!!!

Gluon self energy, $\Pi_{ab}^{\mu\nu}(p)$ – to lowest order in g, $Z_1 = Z_2 = Z_3 = 1$

So:

contributes:

$-i\Pi_{ab}^{\mu\nu}(p) = g_0^2(-1)N_f\dfrac{\delta^{ab}}{2} I_{\mu\nu}(p)$

($g^2\mu^{4-n}$; q-loop; $\text{tr}(t^a t^b)$)

Feed in extra q – take to zero at end....

where $I_{\mu\nu}(p)$ is an integral which is calculated using dimensional regularisation to give:

$\Pi_{\mu\nu}^{ab}(p) = \delta^{ab}N_f\, G_{\mu\nu}(p)\,\Pi(p^2).p^2$

tensor structure: $G_{\mu\nu} = g^{\mu\nu} - \dfrac{p^\mu p^\nu}{p^2}$

$p^2\Pi(p) = \dfrac{g^2}{4\pi}A\left(\dfrac{1}{n-4} + 2(\gamma - \log 4\pi)\right) + \tilde{\Pi}_0\left(\dfrac{p^2}{\mu^2}\right)$ (MS: $\frac{1}{n-4}$ term; $\overline{MS}$: with $2(\gamma-\log 4\pi)$)

Now in Landau gauge, $D_{F\,ab}^{\mu\nu}(p) = -i\delta_{ab}\dfrac{G^{\mu\nu}(p)}{p^2}$

so when calc D_F', tensor structure collapses and get same old result:

$D_{F\,ab}^{\prime\,\mu\nu} = -i\delta_{ab}G_{\mu\nu}(p).\dfrac{1}{p^2(1-\Pi(p))}$

now: the different schemes correspond to what we decide to call Z_3 and what we keep as $\tilde{\Pi}_0$

MS is: $Z_3 = \dfrac{1}{1 - \frac{g^2 A}{4\pi(n-4)}}$ $\overline{MS}$ is: $Z_3 = \dfrac{1}{1 - \frac{g^2}{4\pi}\left[\frac{A}{n-4} + 2(\gamma - \log 4\pi)\right]}$

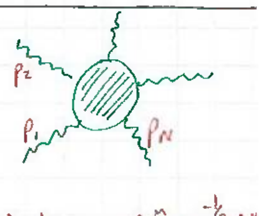
both just absorb the divergence $\frac{A}{n-4}$ but $\overline{MS}$ seems to converge the series faster, but to where? – cannot know unless sum whole series.

Renormalisation Group Equations

Dimensions: Bosons: $M^{\frac{N}{2}(n-2)}$ Fermions: $M^{\frac{N(n-1)}{2}}$

N leg position space green f'ns...

Consider N-gluon (Boson!) leg green f'n:



$G_N(p_1\cdots p_N; g,\mu) = Z_3^{-\frac{N}{2}}G_N^0(p_1\cdots p_N; g_0)$ because $A^\mu = Z_3^{-\frac{1}{2}}A_0^\mu$

now: $\mu\dfrac{d}{d\mu}G_N(p_1\cdots p_N; g,\mu)\Big|_{g_0} = \mu\dfrac{\partial G_N}{\partial\mu}\Big|_{g_0} + \dfrac{\partial G_N}{\partial g}\mu\dfrac{dg}{d\mu}\Big|_{g_0}$ just chain rule....

$-\dfrac{N}{2}\mu\dfrac{d\ln Z_3}{d\mu}.G_N(\;)\Big|_{g_0}$ — $\gamma(g)$; $\beta(g)\dfrac{\partial}{\partial g}$

Callan-Symanzik Equation

$$\boxed{\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \frac{N}{2}\gamma\right)G_N(p;g,\mu) = 0}$$

A.Q.F.T. (7)

Solution of the R.G. eq'n

Massless $\rightarrow \mu$ along is important. Green f'n is homogeneous: $G_N = \mu^{D_N} \underbrace{\tilde{G}_N\left(\frac{\kappa p}{\mu}, g\right)}_{\text{dimless}}$

$$\frac{\partial G_N}{\partial \mu} = D_N \mu^{D_N} \tilde{G}_N\left(\frac{\kappa p}{\mu}\right) + \mu^{D_N} \frac{\partial \tilde{G}}{\partial \mu}\left(\frac{-\kappa p}{\mu^2}\right)$$

$$\kappa \frac{d G_N}{d\kappa} = \mu^{D_N} \frac{\kappa p}{\mu} \tilde{G}_N$$

So: $\left(\underbrace{-\kappa \frac{\partial}{\partial \kappa} + \beta(g) \frac{\partial}{\partial g}}_{} + \gamma_N\right) G_N(\kappa p; g, \mu) = 0$

the engineering dimension $\left(D_N + \frac{N\gamma}{2}\right)$ the anomalous dimension.

rewrite as Δ then: $(\Delta + \gamma_N) G_N(\kappa p; g, \mu) = 0$

this has solution: $G_N(\kappa p; g, \mu) = \underbrace{e^{\int_1^\kappa \frac{d\kappa'}{\kappa'} \gamma_N(\bar{g}(\kappa'))}}_{\text{accumulated field strength renorm!!?!}} G_N(p; \bar{g}(\kappa), \mu)$

suppose $\bar{g} \rightarrow 0$ as $\kappa \rightarrow \infty$ then can get RHS in P.T. !! even if g is big!!

where $\bar{g}(\kappa)$ is the solution to the eq'n $\Delta \bar{g}(\kappa) = 0$ with the boundary condition $\bar{g}(1) = g$

because: $\Delta \bar{g}(\kappa) = 0$ can be solved (i.e.):

$$\kappa \frac{d\bar{g}}{d\kappa} = \beta(g) \frac{d\bar{g}}{dg} \qquad \text{but } \beta(\bar{g}) = \mu \frac{d\bar{g}}{d\mu} = \mu \frac{d\bar{g}}{dg} \frac{dg}{d\mu}$$

$\mu \frac{dg}{d\mu}$; $\beta(g)$

$$\therefore \kappa \frac{d\bar{g}}{d\kappa} = \beta(\bar{g}(\kappa))$$

$$\left[\kappa \frac{\partial}{\partial \kappa} - \beta(g) \frac{\partial}{\partial g}\right] \bar{g}(\kappa) = 0 \quad \text{is the same as:} \quad \kappa \frac{d\bar{g}(\kappa)}{d\kappa} = \beta(\bar{g}(\kappa))$$

this has the solution: $\log \kappa = \int_g^{\bar{g}(\kappa)} \frac{d\bar{g}(\kappa')}{\beta(\bar{g}(\kappa'))}$

if differentiate w.r.t. g, get first eq'n back...

So: the sign of $\beta(\bar{g}(\kappa))$ determines the rate of change of running coupling w.r.t. mom scale...

generally: $g = Z_1 Z_2 Z_3^{1/2} g_0 \mu^{\frac{n}{2}-2}$

then $\beta(g) = \mu g \frac{\partial}{\partial \mu} \log\left(Z_1 Z_2 Z_3^{\frac{1}{2}}\right) + g\left(\frac{n}{2} - 2\right)$

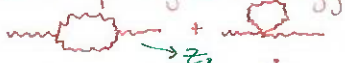
now in MS or $\overline{MS}$ each Z has a pole

$\therefore$ suppose: $Z_1 Z_2 Z_3^{1/2} = 1 + \frac{\lambda g^2}{n-4} + O(g^4)$

then $\beta(g) = \frac{2\lambda g^2 \beta(g)}{n-4} + g\left(\frac{n}{2} - 2\right) + O(g^5)$

So as $n \rightarrow 4$, get $\beta(g) = \lambda g^3 + \ldots$

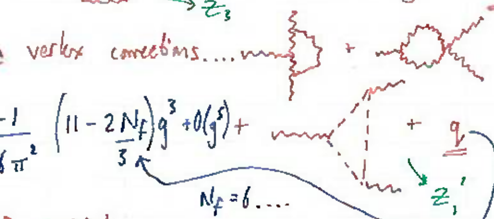
if include all fermion loops in gluon self energy aswell as gluon loops ie



$+$ ghost $+$ q

Z_3

and include gluon vertex corrections....



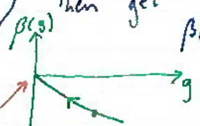
$+$ q

Z_1'

then get:

$$\beta(g) = \frac{-1}{16\pi^2}\left(11 - \frac{2N_f}{3}\right) g^3 + O(g^5) + \ldots$$

$N_f = 6 \ldots$



zero coupling fixed point $\rightarrow$ as $\kappa\uparrow$, $\bar{g}(\kappa)\downarrow$ until $\bar{g}(\kappa) \rightarrow 0$, get UV fixed point.....

Possibilities:

start.... ; start

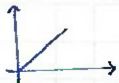
① $\bar{g} \rightarrow 0$ as $\kappa\uparrow$
② $\bar{g} \rightarrow \infty$ as $\kappa\downarrow$

① $\bar{g} \rightarrow 0$ as $\kappa\uparrow$
② $\bar{g} \rightarrow g^*$ as $\kappa\downarrow$

① $\bar{g} \rightarrow \infty$ as $\kappa\uparrow$
② $\bar{g} \rightarrow g^*$ as $\kappa\downarrow$

① is assyptotic freedom.

In QED only have fermion loop terms so β is $+$ve



$\therefore$ CERN uses $\frac{1}{128}$!!....

Effective Action

let $-iW[J] = \log Z[J]$ - then $W[J]$ generates all connected correlation functions (Green f'ns $G(x_1 \cdots x_n)$)

can write $W[J] = \sum_N \frac{i^N}{N!} \int d^4x_1 \ldots d^4x_N \, G(x_1, \cdots x_N) \underbrace{J(x_1) \cdots J(x_N)}$ coz $\frac{\delta}{\delta J(x_i)}$ can hit all of these

now $\frac{-i\delta W[J]}{\delta J(x)} = \bar{\Phi}(x)$

and when $J = 0$, $\bar{\Phi}$ = vacuum expectation val, $\langle 0 | \phi(x) | 0 \rangle = 0$ unless spon. sym break... so have trans inv.

Now do Legendre Transformation:

let $\underbrace{\Gamma[\bar{\Phi}(x)]}_{} = W - i \int d^4x' \underbrace{J(x') \bar{\Phi}(x')}_{h \cdot m}$

cf. Gibbs free energy, Helmholtz

Then: $-i\frac{\delta \Gamma}{\delta \bar{\Phi}(x)} = \int d^4y \left(\frac{-i\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \bar{\Phi}(x)} - \int d^4x' \left[J(x') \frac{\delta \bar{\Phi}(x')}{\delta \bar{\Phi}(y)} \frac{\delta \bar{\Phi}(y)}{\delta \bar{\Phi}(x)} + \bar{\Phi}(x') \frac{\delta J(x')}{\delta J(y)} \frac{\delta J(y)}{\delta \bar{\Phi}(x)}\right]\right)$

$\delta^{(4)}(x'-y)$; $\delta^{(4)}(x-y)$; $\delta^{(4)}(x'-y)$

cancel

So: $\boxed{\frac{i\delta\Gamma}{\delta\bar{\Phi}(x)} = J(x)}$ Physic is $J = 0 \therefore \Gamma$ is an effective potential !!

Now, similarly to here, define $\Gamma(x, \cdots x_N)$ as:

$$\Gamma[\bar{\Phi}] = \sum_N \frac{i^N}{N!} \int d^4x_1 \cdots d^4x_N \, \Gamma(x_1, \cdots x_N) \, \bar{\Phi}(x_1) \cdots \bar{\Phi}(x_N)$$

But what really is $\Gamma(x_1, \cdots x_N)$? look at $\frac{\delta \bar{\Phi}(x)}{\delta \bar{\Phi}(y)}$

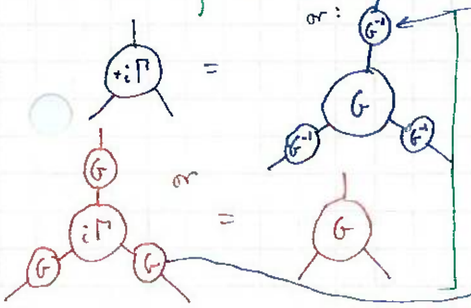
$$\delta^{(4)}(x-y) = -i \int d^4z \underbrace{\frac{\delta^2 W[J]}{\delta J(z) \delta J(x)}}_{G(x-z)} \underbrace{\frac{\delta J(z)}{\delta \bar{\Phi}(y)}}_{\frac{\delta^2 \Gamma}{\delta \bar{\Phi}(y) \delta \bar{\Phi}(x)} = \Gamma(z-y)}$$

ie $\delta^{(4)}(x-y) = \int d^4z \, G(x-z) \Gamma(z-y)$

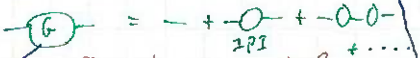
i.e. $\boxed{i\Gamma(p) = p^2 - m_0^2 - \Pi(p)}$ - self energy! ie 1PI 2pt f

Use $\bar{\Phi}(x) = \frac{-i\delta W[J]}{\delta J(x)}$ to evaluate higher derivatives:

$$i\Gamma(x, y, z) = \int d^4x' d^4y' d^4z' \, G(x', y', z') \Gamma(x, x') \Gamma(y, y') \Gamma(z, z')$$

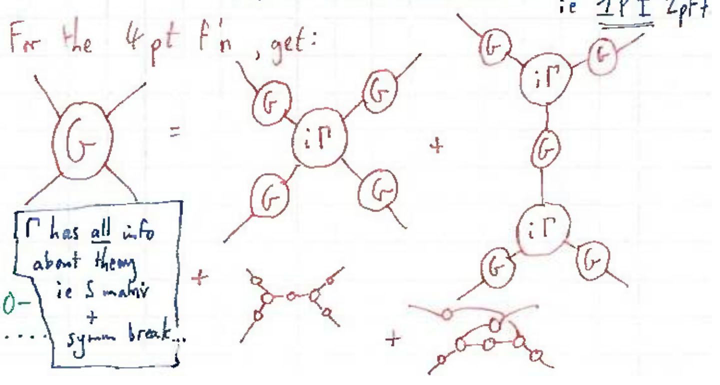


ie $i\Gamma(x,y,z)$ is the exact 3 point f'n but without the poles - it is 1PI or amputated - all self energy bits have been taken off ie.



so Γ is the proper vertex?

For the 4 pt f'n, get:



Γ has all info about theory ie S matrix + symm break...

A.Q.F.T. ⑧

Ward-Takahashi Identity

- Direct consequence of the Gauge Symmetry.

For Q.E.D.

$$Z[J^\mu,\sigma,\bar\sigma] = \int dA^\mu d\psi d\bar\psi \, e^{i\int d^4x\left(\mathcal{L}_{QED} + \mathcal{L}_{GF} + J\cdot A + \bar\sigma\psi + \bar\psi\sigma\right)}$$

$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(\partial\cdot A)^2$ in covariant gauges; the whole exponent is $\mathcal{L}_{old}$.

Ghosts have been integrated out - they don't couple to photons and so are contained in defn of $d\phi$ as an ∞ const like vac. bubbles....

(Generalise to non-abelian $\longrightarrow$ Slavnov Taylor Identities....)

Make an infinitesimal gauge transformation:

$\psi \to \psi - ie\Lambda(x)\psi$ and $A^\mu \to A^\mu + \partial^\mu\Lambda(x)$

$\mathcal{L}_{QED}$ is gauge invariant but $\mathcal{L}_{GF}$ + the rest aren't:

$$\to \left[-\frac{1}{\alpha}(\partial\cdot A)(\partial^2\Lambda)\right] - \left[\partial\cdot J + ie(\bar\sigma\psi + \bar\psi\sigma)\right]\Lambda$$

(first term: parts twice $\to \Lambda\partial^2(\partial\cdot A)$; second term: parts)

Then expand exponential to $O(\Lambda)$, get:

$$Z^\Lambda[J^\mu,\sigma,\bar\sigma] = \int \frac{dA^\mu d\psi d\bar\psi}{d\phi}\left(1 + \{\ \}\Lambda\right)e^{i\int d^4x\,\mathcal{L}_{old}}$$

where the integrand is $F[\phi_\Lambda]$ and

$$\left\{-\frac{1}{\alpha}\partial^2(\partial\cdot A) - \partial\cdot J - ie(\bar\psi\sigma - \bar\sigma\psi)\right\}$$

now: integration measure is gauge invariant:

$$\int d\phi\, F[\phi] = \int d\phi_{-\Lambda}\, F[\phi]$$

but can relabel variable $\phi_{-\Lambda} \to \phi$

$$\therefore \int d\phi\, F[\phi] = \int d\phi\, F[\phi_\Lambda]$$

So, $\{\ \} = 0$

But can use effective action idea:

let $i\Gamma[A,\psi,\bar\psi] = W[J,\sigma,\bar\sigma] - i\int d^4x\,(J\cdot A + \bar\sigma\psi + \bar\psi\sigma)$

So $\partial\cdot J = \partial_\mu \frac{\delta\Gamma}{\delta A_\mu}$, $\bar\psi\sigma = \bar\psi(x)\frac{\delta\Gamma}{\delta\bar\psi(x)}$, $\bar\sigma\psi = \frac{\delta\Gamma}{\delta\psi(x)}\psi(x)$

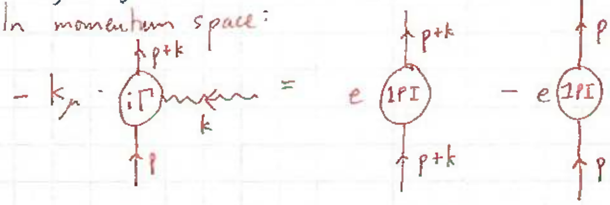
then $\{\ \} = 0$ means that:

$$0 = -\frac{1}{\alpha}\partial^2(\partial\cdot A) + \partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)} + e\frac{\delta\Gamma}{\delta\psi(x)}\psi(x) - e\bar\psi(x)\frac{\delta\Gamma}{\delta\bar\psi(x)}$$

now apply $\frac{\delta^2}{\delta\psi(y)\delta\bar\psi(z)}$ and set $A,\psi,\bar\psi = 0$, to get:

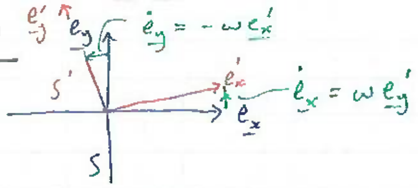
$$\partial_\mu\frac{\delta^3\Gamma}{\delta A_\mu(x)\delta\psi(y)\delta\bar\psi(z)} = e\,\delta^{(4)}(x-y)\frac{\delta^2\Gamma}{\delta\bar\psi(z)\delta\psi(x)} - e\,\delta^{(4)}(x-z)\frac{\delta^2\Gamma}{\delta\bar\psi(x)\delta\psi(y)}$$

In momentum space:



which tends to Ward Identity as $k \to 0$.

1B Dynamics ①



Ficticious Forces

Position vector $\underline{r} = r_a \underline{e}_a = r_a' \underline{e}_a'$

$\dot{\underline{r}} = \dot{r}_a' \underline{e}_a + \underline{\omega} \times \underline{r}$ where $\underline{\omega} = \omega \underline{e}_z$

Do twice: $\ddot{r}_a e_a = \left(\ddot{\underline{r}} = \ddot{r}_a' \underline{e}_a' + 2\underline{\omega} \times (\dot{r}_a' \underline{e}_a') + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \right) \times m$

- $\ddot{r}_a' e_a'$: force as meas'd in acc'ing frame S'
- $\dot{r}_a' e_a'$: vel as meas. in S' — coriolis
- $\omega \times (\omega \times r)$: centrifugal
- $\ddot{r}_a e_a$: "real" force, as measured in inertial frame S

So $\underline{F}_{rot} = F_{\text{"real"}} - 2m\underline{\omega} \times \dot{\underline{r}}\big|_{S'} - m\underline{\omega} \times (\underline{\omega} \times \underline{r})$

Inertia tensor with cartesian basis:

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \sum_i m_i \begin{pmatrix} r_i^2 - x_i^2 & -xy & -xz \\ -xy & r^2 - y^2 & -yz \\ -xz & -yz & r^2 - z^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$\underline{J}$, $\underline{\underline{I}}$, $\underline{\omega}$ — diagonalise! → principle axes etc

$E_k = T = \frac{1}{2} \omega^T I \omega \rightarrow$ ellipsoid.

Euler's Equations

$\underline{G} = \dot{\underline{L}}_{space} = \dot{\underline{L}}_{body} + \underline{\omega} \times \underline{L}$

($\dot{L}_{space} \uparrow G$; $\dot{L}_{body} \uparrow \underline{\underline{I}}\,\dot{\underline{\omega}}$)

w.r.t. princ. axes → $G_1 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2)$ + perms

free precession for a symmetric top: $G = 0$, $I_1 = I_2$

$$\left.\begin{aligned} I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_1 - I_3) \\ I_2 \dot{\omega}_2 &= -\omega_1 \omega_3 (I_1 - I_3) \end{aligned}\right] \rightarrow \text{SHM, freq } \Omega_b = \left(\frac{I_1 - I_3}{I_1}\right)\omega_3$$

Elasticity

Poisson: $\frac{\Delta w}{w} = \frac{\Delta h}{h} = -\sigma \frac{\Delta L}{L}$

Hookes: $\frac{F}{A} = Y \frac{\Delta L}{L}$

→ relate to bulk modulus: uniform hyd. press ⟷ squish in x, y, z dirns

added together: $\frac{\Delta L}{L} = -\frac{p}{Y} + 2\sigma \frac{p}{Y} \Rightarrow B = \frac{Y}{3(1-2\sigma)}$ ∴ $\sigma < \frac{1}{2}$, otherwise unstable eq.

(Bulk: $\frac{F}{A} = -B \frac{\Delta V}{V}$)

Shear: $n = \frac{\tau}{\theta} = \frac{Y}{2(1+\sigma)}$

Stress tensor: $\underline{F} = \underline{\underline{\tau}}\, \underline{A}$

pure shear ⟷ traceless

eg $\begin{pmatrix} \tau & \cdot & \cdot \\ \cdot & -\tau & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$

GAUGE THEORY GRAVITY

Dirac Eq'n: $\nabla\psi I\sigma_3 = m\psi\gamma_0$ → **Observables**: $\langle\phi|\psi\rangle$ or **spinor eq'ns** $\psi = \psi_1 + \psi_2$ — these are locally gauge inv.

globally gauge inv.

→ so let's make the eq'n locally gauge inv.

So ∇ becomes $D = \nabla$ + extra bit.

Basic idea: eg spinor eq'n $\psi_1(x) = \psi_2(x)$. Pos'n g. tr'm doesn't change any physical content. same with rot'n g. tr'ms.

Observables: $J = \psi\gamma_0\tilde{\psi}' \to J' = RJ\tilde{R}$

- it does change but eq'ns invariant

∴ absolute dir'n not observable.

So: $= \partial_a(a\cdot\nabla\psi + \frac{1}{2}\psi\Omega(a))$ (contracted)

$D'(\psi') = (D\psi)'$ ie $D'(\psi R) = D\psi R$

$\Rightarrow \Omega'(a) = \tilde{R}\Omega(a)R - 2\tilde{R}a\cdot\nabla R$

Position Gauge Field $x \to f(x)$ $\psi(x') = \psi'(x)$

now $\nabla\psi'(x)$ transforms how?

$\partial_a a\cdot\nabla\psi(f(x)) = \lim_{\varepsilon\to 0}\frac{\psi[f(x+\varepsilon a)] - \psi[f(x)]}{\varepsilon}$

$= \lim_{\varepsilon\to 0}\frac{\psi[f(x)+\varepsilon a\cdot\nabla f(x)] - \psi[f(x)]}{\varepsilon}$

$= \underline{f}(a)\cdot\nabla_{x'}\psi(x') = a\cdot\bar{f}(\nabla_{x'})\psi(x')$

So $\nabla_x = \bar{f}(\nabla_{x'})$ now consider $\bar{h}(\nabla_x)\phi(x) = A(x)$ (gauge field...)

Pos'n g. tr'm; lhs: $\bar{h}'_x(\nabla_x)\phi'(x) = \bar{h}'_x(\bar{f}(\nabla_{x'}))\phi(x')$

$= A(x')$? only if: $\boxed{\bar{h}'_x(a) = \bar{h}\bar{f}^{-1}_x(a)}$

take inside cos scalar: $\bar{h}'_x\bar{f}_x(\nabla_{x'}\phi_{x'})$, $\bar{h}_{x'}^{-1}(A_{x'})$

NB position dependence!

So now: $\bar{h}(\nabla)\psi I\sigma_3 = m\psi\gamma_0$ is position gauge covariant.

Rotation Gauge Field

rot. g tr'm for spinors: $\psi' = R\psi$ (D_a has Ω on different side)

as above, $\nabla \to D$ ie $\bar{h}(\nabla) = \bar{h}(\partial_a)a\cdot\nabla$ — now cf. QED: $D_a = a\cdot\nabla + \frac{1}{2}\Omega(a)$

now rotate lhs of Dirac eq'n:

$\bar{h}(\partial_a)D_a\psi I\sigma_3 \to \bar{h}'(\partial_a)D'_a(R\psi)I\sigma_3$

$= \bar{h}'(\partial_a)RD_a(\psi)I\sigma_3$ — covariance of D_a ie $D'_a\psi' = RD_a\psi$

only true if $= mR\psi\gamma_0$ (rhs)

$\bar{h}'(\partial_a)R = R\bar{h}(\partial_a) \Rightarrow \boxed{\bar{h}'(a) = R\bar{h}(a)\tilde{R}}$

Covariant Derivatives of Observables → ie $A = \psi\Gamma\tilde{\psi}$ where $\Gamma = \gamma_0, \gamma_3, I\sigma_3$ etc.

From spinor tr'm laws, get these for A: $A'(x) = A(x')$, $A'_x = RA_x\tilde{R}$ ← covariant tr'ms.

now $a\cdot\nabla A = (a\cdot\nabla\psi)\Gamma\tilde{\psi} + \psi\Gamma(a\cdot\nabla\tilde{\psi})$

$\mathcal{D}_a A = D_a\psi\Gamma\tilde{\psi} + \psi\Gamma(D_a\psi)^\sim$

$= a\cdot\nabla A + \Omega(a)\times A$

(and $\mathcal{D} = \bar{h}(\partial_a)\mathcal{D}_a$)

it is still a deriv as satisfies Liebniz...

Rotation Gauge Field Strength

$[D_a, D_b]\psi = \frac{1}{2}\mathcal{R}(a\wedge b)\psi$ — coz commutator is symmetric.

where $\mathcal{R}(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega_a\times\Omega_b$

actually through: $\mathcal{R}(a\wedge b + c\wedge d) = \mathcal{R}(a\wedge b) + \mathcal{R}(c\wedge d)$ — nonlinear field eq'ns...

can write $\mathcal{R}(B)$

now $[D'_a, D'_b]\psi' = R[D_a, D_b]\psi = \frac{1}{2}\mathcal{R}'(a\wedge b)R\psi$

So $\mathcal{R}'(a\wedge b) = R\,\mathcal{R}(a\wedge b)\tilde{R}$

Displacement Gauge Field Strength

Consider: $[a\cdot\bar{h}(\nabla), b\cdot\bar{h}(\nabla)]\psi = (b\wedge a)\cdot[\bar{h}(\nabla)\wedge\bar{h}(\nabla)]\psi$ using standard result.

$\bar{h}(\nabla)\wedge\bar{h}(\nabla)\phi = \bar{h}(\dot{\nabla})\wedge\dot{\bar{h}}(\nabla\phi)$ from chain rule and $\nabla\wedge\nabla\phi = 0$.

let $\nabla = \bar{h}^{-1}(a)$ for some reason!

then $S(a) = \bar{h}(\dot{\nabla})\wedge\dot{\bar{h}}\bar{h}^{-1}(a) = \bar{h}(\nabla\wedge\bar{h}^{-1}(a))$ using $\bar{h}(\dot{\nabla})\wedge\bar{h}\dot{\bar{h}}^{-1}(a) = 0$.

↑ bivector.

Covariant Field Strengths: Riemann Tensor

under displ: $\mathcal{R}(B) \to \mathcal{R}'_x(B) = \mathcal{R}_{x'}(\underline{f}(B))$

now: $\bar{h} \to \bar{h}\bar{f}^{-1}$

so $\underline{h} \to \underline{f}^{-1}\underline{h}$

So $\boxed{\mathsf{R}(B) = \mathcal{R}\underline{h}(B)}$ is covariant

→ ie $\mathsf{R}(B) \to R\,\mathsf{R}(\tilde{R}BR)\tilde{R}$ under rot.

$\mathsf{R}(B;x) \to \mathsf{R}(B;x')$ under displ.

← Riemann Tensor

eg Kerr: $\mathsf{R}_K(B) = \frac{-M}{2(r+IL\cos\theta)^3}(B + 3\sigma_r B\sigma_r)$

Homogeneous, Isotropic Cosmology: $\mathsf{R}_{cos}(B) = 4\pi(\rho + p)B_t e_t - \frac{1}{3}(8\pi\rho + \Lambda)B$

Astrophysical Fluid Dynamics ①

Basic Fluid Dynamics - Ideal Gas

- Low Densities: ISM diffuse clouds $\left[n_H \sim 10 - 10^3 \text{ cm}^{-3},\ T \sim 50 - 150\text{K} \right]$ or molecular clouds $\left[n_{H_2} \sim 10^3 - 10^6 \text{ cm}^{-3},\ T \sim 3 - 10\text{K} \right]$ } ideal

→ Ideal gas: $pV_m = \frac{p}{\rho} = \frac{kT}{\mu}$ or $p = nkT$. (V_m: specific vol)

Also
$$c_s^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \qquad \gamma = \frac{c_p}{c_v} \qquad c_p - c_v = \frac{k_B}{\mu}$$

and:
$$u_m = c_v T = \frac{pV_m}{\gamma - 1} = \frac{c_s^2}{\gamma(\gamma-1)} \quad \text{spec. energy density}$$
$$h_m = c_p T = \frac{c_s^2}{(\gamma - 1)} \quad \text{spec. enthalpy density.}$$
$$S_m = c_v \log\left(\frac{p}{\rho^\gamma}\right) \qquad \text{adiabatic} \Rightarrow p\rho^{-\gamma} = \text{const}.$$

Euler's Equation (zero viscosity)

Newton's 2nd Law for fluid element ΔV:
$$\int_{\Delta V} \rho \frac{D\underline{v}}{Dt} dV = \int_{\Delta V} \underline{f}\, dV - \int_S p\, d\underline{S}$$
($\underline{f}$: force per unit vol on ΔV; p: pressure in ΔV)

then:
$$\rho \frac{D\underline{v}}{Dt} = \underline{f} - \underline{\nabla} p \quad + \eta \nabla^2 \underline{v} - \frac{2}{3}\eta \underline{\nabla}(\underline{\nabla}\cdot\underline{v})$$

If add viscosity, get the Nav. Stokes eq'n....

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}$$

often $\underline{f} = -\rho \underline{\nabla}\Phi_g$

gives rise to $Q = \eta |\underline{\nabla}\times\underline{v}|^2$

Energy Conservation:
$$\int_{\Delta V} Q\, dV = \int_S \text{Flux}\cdot d\underline{S} + \frac{d}{dt}\int \text{energy density}\cdot dV$$
$$\text{Flux} = \rho\underline{v}\left(\tfrac{1}{2}v^2 + u_m + \frac{p}{\rho}\right) = \rho\underline{v}\left(\tfrac{1}{2}v^2 + h_m\right)$$
$$\text{energy density} = \tfrac{1}{2}\rho v^2 + \rho u_m$$

So
$$Q = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho u_m\right) + \underline{\nabla}\cdot\left(\rho\underline{v}\left(\tfrac{1}{2}v^2 + h_m\right)\right)$$

Conservation of mass for fluid element ΔV:
$$\frac{\partial}{\partial t}\int_{\Delta V}\rho\, dV + \int_{\Delta V}\rho\underline{v}\cdot d\underline{S} = 0 \quad \text{then} \quad \frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v}) = 0$$

Bernoulli's Equation

$$\tfrac{1}{2}v^2 + h_m + \Phi_g = \text{const}$$

integrate Euler for steady state, replace $\underline{v}\cdot\underline{\nabla}\underline{v}$ .....

along a streamline for adiabatic flow, → then $dh_m = \frac{dp}{\rho}$

Steady Gas Flow

compressible, maybe supersonic.

Bernoulli: $\tfrac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \text{const}$.

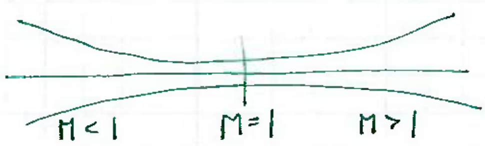
Euler, along streamline: $v\,dv = -\frac{dp}{\rho} = -c_s^2\frac{d\rho}{\rho}$ (by def'n.)

So mass flux, $\frac{d}{dv}(\rho v) = \rho\left(1 - \frac{v^2}{c_s^2}\right)$ ie max when $v = c_s$

Mach no. $M = \frac{v}{c_s}$. $V_{max} = \sqrt{2h_m^{\text{stationary}}} = \left(\frac{2}{\gamma - 1}\right)^{1/2} c_{\text{stationary}}$ (~ 1)

ie. mass conser. $d(\rho v A) \Rightarrow \frac{dA}{A} = -\frac{d(\rho v)}{\rho v} = (M^2 - 1)\frac{dv}{v}$

So get



Equilibria of self gravitating gaseous bodies

Only consider static eqm, radiation. Ignore convection, conduction

Basic equations:

Euler: $\frac{dp}{dr} = -\frac{GM\rho}{r^2}$ ① Mass: $\frac{dM}{dr} = 4\pi r^2\rho$ ②

Radiation: $\frac{dL}{dr}\left(= \frac{d}{dr}(4\pi r^2 F_r)\right) = 4\pi r^2\rho\varepsilon$ ③

and $F_r = -\frac{c}{3\kappa\rho}\frac{\partial}{\partial r}(aT^4)$ ④ radiation diffusion.

Isothermal Sphere:

⇒ Euler + Mass eq'ns ~~are~~ is unchanged No!!!

$p = \rho\frac{kT}{\mu}$ ∴ $p' = \frac{kT}{\mu}\rho'$ (σ^2) as isoth. ⇒ ....

$$\frac{d\rho}{dr} = -\frac{GM}{\sigma^2}\frac{\rho}{r^2}$$

one solution: (Singular Isoth. sph)
$$\rho = \frac{\sigma^2}{2\pi G r^2}$$

must be consistent with ②....

Looks like $p, \rho \to \infty$ at centre

core must have a density ρ_0 say, then core radius r_0 is $\left(\frac{9\sigma^2}{4\pi G\rho_0}\right)^{1/2}$.

Astrophysical Fluid Dynamics ②

Virial Theorem

Take Euler eq'n and mass eq'n:

$$4\pi r^3 dp = -4\pi r GM\rho\, dr = -\frac{GM}{r} dM$$

and integrate:

$$\int_{p_0}^{p_c} 3V dp = 3\Big[pV\Big]_{0,p_c}^{V,p_0} - 3\int_0^{V_0} p\,dV = -\int_0^{M_0} \frac{GM(r)dM}{r}$$

($3[pV]$ term: $4\pi r_0^3 p_0$; $p\,dV$: $\frac{p}{\rho} dM$; RHS: Grav. P.E., Ω)

$$\boxed{3\int_0^{M_0} \frac{p}{\rho} dM + \Omega = 4\pi r_0^3 p_0}$$

(2×) K.E. term, P.E., ~0

If assume $p_{rad} \sim 0$ and ideal gas, $\frac{p}{\rho} = \frac{kT}{\mu}$

get: $\frac{3k}{\mu}\int_0^{M_0} T dM \equiv \frac{3k\bar{T}}{\mu} M_0$ ← $\frac{M_0}{\mu} = N$!

$\bar{T}$ – mass averaged temp.

$\Rightarrow T_c$ for sun is $\geq 2\times10^6$ K.

Cloud stability

Assume isothermal then: $3(\gamma-1)u_m M_0 + \Omega \;\; (<\text{ collapse},\ =\text{ stable},\ >\text{ expand})\;\; 4\pi r_0^3 p_0$

Spherical accretion

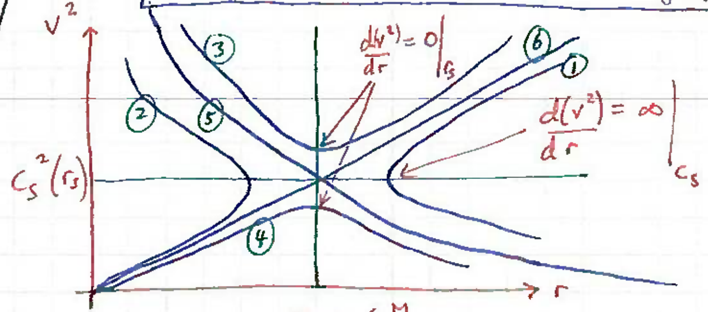
Euler: $\frac{\partial v}{\partial t} + v\frac{dv}{dr} + \frac{1}{\rho}\frac{dp}{dr} + \frac{GM}{r^2} = 0$ —— (*)

($\frac{\partial v}{\partial t} \to 0$; first two terms: $\frac{Dv}{Dt}$; ∇p; $-\nabla\Phi_g$)

Mass cons: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho \underline{v}) = 0 \Rightarrow \rho v r^2 = \text{const} = -\frac{\dot{M}}{4\pi}$ (mass flow rate)

elim $\frac{dp}{dr}$ using $c_s^2 = \frac{dp}{d\rho}$ and [mass cons] to get: ($c_s = c_s(r)$....)

$$\boxed{\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{d(v^2)}{dr} = -\frac{GM}{r^2}\left[1-\frac{2c_s^2 r}{GM}\right]}$$



$r_s = \frac{GM}{2c_s^2}$

Which is the physical solution? ①, ② unphysical, $\frac{d(v^2)}{dr} = \infty$....

③ ...? ④ subsonic ∴ no good ($\rho(0) = \infty$)

⑥ wind ⑤ accretion ???

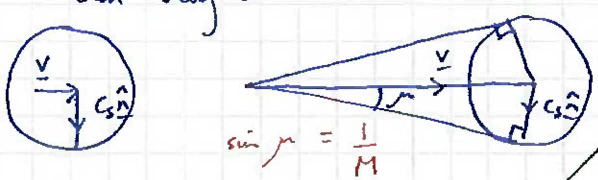
Shocks / Supersonic Flows.

"Disturbance Propagation"

i.e. change in pressure, mass outflow (effect of piston), fluid instability.

$K(\underline{x}, t)$ → FT → $K(\underline{k}, \omega)$ → all harmonics travel at c_s

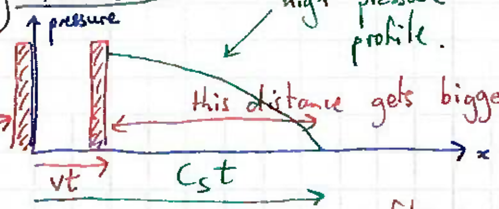
- can only affect "Mach Cone":



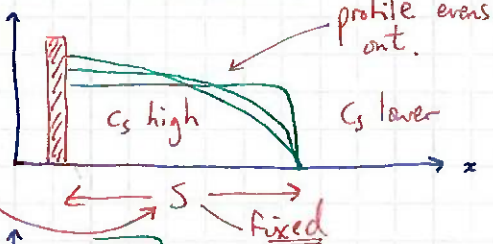
$\sin\mu = \frac{1}{M}$

Instability Formation

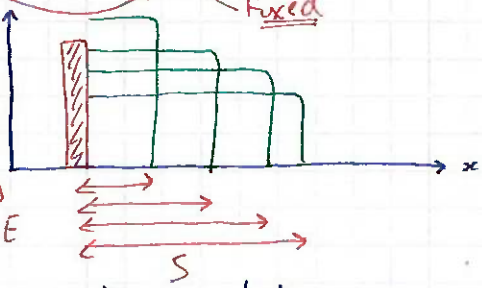
Subsonic:



Near-critical: distance S remains ~ const. but c_s inside is $> c_s$ outside!



Super sonic
- S gets smaller
pressure increases
→ unstable
→ all energy goes into thermal KE
→ BANG!



zero viscosity / conduction approx must break down.

Shock Conditions

i.e. matching conditions at shock boundary:

- Shock itself is dissipative in that it changes KE into heat (no conserve energy....)

$v_1\ \rho_1\ p_1$ | $v_2\ \rho_2\ p_2$ → high ρ → subsonic....

upstream → shock ← downstream

In shock rest frame:

Conserve: mass flow:

$$\rho_1 v_1^x = \rho_2 v_2^x$$

Conserve Mom. Flow:

$$p_1 + \rho_1 v_1^{x\,2} = p_2 + \rho_2 v_2^{x\,2}$$
$$\rho_1 v_1^x v_1^y = \rho_2 v_2^x v_2^y$$
$$\rho_1 v_1^x v_1^z = \rho_2 v_2^x v_2^z$$

Conserve enthalpy flow

$$\rho_1 v_1^x\left(\tfrac{1}{2}v_1^2 + h_1\right) = \rho_2 v_2^x\left(\tfrac{1}{2}v_2^2 + h_2\right)$$

Astrophysical Fluid Dynamics ③

Normal Adiabatic Shock

take eq'ns, express in terms of γ and Mach no using ideal law: $p = \frac{\gamma \rho V_m}{(\gamma - 1)}$

For $M_1^2 \gg 1$

→ get $\frac{\rho_2}{\rho_1} \simeq \frac{\gamma+1}{\gamma-1} \sim 4$

$$\frac{p_2}{p_1} \simeq \frac{2\gamma M_1^2}{\gamma+1} \sim \frac{5}{4} M_1^2$$

$$\frac{T_2}{T_1} \simeq \frac{2\gamma(\gamma-1) M_1^2}{(\gamma+1)^2} \sim \frac{5}{16} M_1^2$$

S.N. Remnants.

- Like universe in GR, express in terms of single scale factor and co-moving coords

-2 approximation regimes:

Phase I : Mass of ejecta (M_0) >> swept up mass (from ISM)

$M_0 \sim M_\odot$ and $E_0 \sim 10^{44}$ J

$r = \lambda R(t)$

$v(r,t) = \lambda \dot{R}$ (dim'less)

$\rho(r,t) = \frac{M_0}{R^3} \hat{\rho}(\lambda)$

$p(r,t) = \frac{U(t)(\gamma-1)\hat{p}(\lambda)}{R^3}$

ideal gas → adiabatic ⇒ $U(t) = E_0 \left(\frac{R_0}{R}\right)^{3(\gamma-1)}$

N2L: $\rho \frac{d}{dt}(\lambda \dot{R}) = -\frac{d}{dr} p$

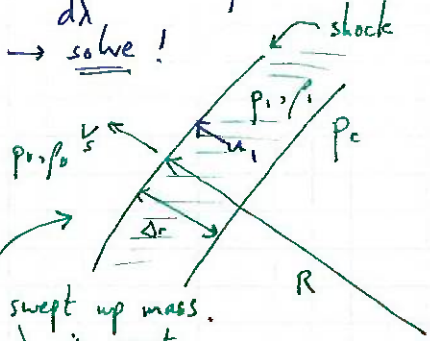
$\Rightarrow M_0 R \ddot{R} \lambda \hat{\rho}(\lambda) = -U(t)(\gamma-1)\frac{d\hat{p}}{d\lambda}$

seperable

$$M R \ddot{R} = k(\gamma-1) U(t)$$

$$\frac{d\hat{p}}{d\lambda} = -k \lambda \rho(\lambda)$$

→ solve !



Phase II Mass of ejecta (M_0) << swept up mass.
Now ρ_0 (external density) is const.

$\lambda = \left(\frac{\rho_0}{E_0}\right)^{1/5} \frac{r}{t^{2/5}}$ is dimensionless. At shock, $\lambda = \lambda_s$ and $r = R = \lambda_s \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5}$.

Want $R(t)$: use N2L for swept up shell:

$$\frac{d}{dt}(M u_1) =$$